

Generalizing the Tomboulis-Yaffe Inequality to $SU(N)$ Lattice Gauge Theories and General Classical Spin Systems

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Abstract

We extend the inequality of Tomboulis and Yaffe in $SU(2)$ lattice gauge theory (LGT) to $SU(N)$ LGT and to general classical spin systems, by use of reflection positivity. Basically the inequalities guarantee that a system in a box that is sufficiently insensitive to boundary conditions has a non-zero mass gap. We explicitly illustrate the theorem in some solvable models. Strong coupling expansion is then utilized to discuss some aspects of the theorem. Finally a conjecture for exact expression to the off-axis mass gap of the triangular Ising model is presented. The validity of the conjecture is tested in multiple ways.

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1 Introduction

In this paper we generalize the inequality proved originally by Tomboulis and Yaffe in $SU(2)$ gauge theories [1] to $SU(N)$ gauge theories with general N and also to a wide range of classical spin systems. To make this paper readable for those working on spin systems, we give in this section an elementary introduction to studies of quark confinement in lattice gauge theories (LGT) with an emphasis on interrelations between concepts in spin systems and those in gauge theories.

That a specific kind of defect could be responsible for determining a phase structure of a statistical system is appreciated as a quite useful idea in wide areas of modern physics. It has a long history, possibly dating back to R. Peierls' argument on the Ising model [2]. In as early as 1944 L. Onsager, in his famous paper on the exact solution of the two-dimensional square Ising model, calculated what he called a 'boundary tension' (the free energy per unit length of a domain wall separating two regions of opposite magnetic order) and found that it is zero above and nonzero below the critical temperature [3]; hence magnetization is not the only quantity that can characterize the phase structure of the system. The history after Onsager clearly tells us the significance of understanding how defects, or dislocations, induce the volatility of the order parameter: for instance the seminal work of Kosterlitz and Thouless [4] made understanding the infinitely smooth phase transition of the XY -model possible by adopting the chemical potential of vortices as an order parameter.

Such an idea was imported into the studies of $SU(N)$ gauge theories ingeniously by 't Hooft [5], Mack and Petkova [6] and several others [7]. Remember that in spin models, the system is said to be in a disordered phase if the two-point correlation function decays exponentially making the correlation length finite; otherwise the system is said to be either in an ordered phase or in a Kosterlitz-Thouless-type phase. In parallel, a non-Abelian gauge theory is said to be in a confining phase if the expectation value of a Wilson loop decays exponentially with the area it spans; otherwise the system is said to be either in the Higgs phase or in the Coulomb (or massless) phase. The $SU(N)$ gauge theory with no matter field has been believed to be in a confining phase for entire values of coupling constant ($SU(3)$ is of special importance as it is supposed to be the true theory of strong interactions in nature, where quarks have never been directly observed in experiments). What is mysterious is that a rigorous proof of confinement is still missing in spite of a tremendous amount of work dedicated to this issue so far. However, according to the scenario(s) pioneered by 't Hooft, Mack, Petkova and others [7], the rapid decay of a Wilson loop expectation value might be attributable to a percolation of center vortices. It is an object of co-dimension 2 (thus it is a loop in 2+1 spacetime and a closed surface in 3+1 spacetime). A rough explanation of their appearance is as follows: in pure $SU(N)$ gauge theory, all the fields belong to the adjoint representation of $SU(N)$, so that the actual gauge group is $SU(N)/Z_N$ rather than $SU(N)$. $\Pi_1(SU(N)/Z_N) = Z_N$ means that the system has a line defect associated to each element of Z_N , which is denominated as a 'center vortex', or a 't Hooft loop'. If a vortex associated to $z \in Z_N$ wraps around the Wilson loop, the latter is multiplied by a factor z . Then the appearance of infinitely many center vortices piercing the Wilson loop randomly *with no mutual correlation* can efficiently disorder the value of the Wilson loop, resulting in an exponential suppression of the expectation value for larger loops follows.¹

Roughly speaking, the formulation of 't Hooft concerns a macroscopically large center vortex wrapping around the periodic lattice, ensuring its presence by imposing a twisted boundary condition on the lattice. This procedure is essentially tantamount to imposing an anti-periodic condition to produce a domain wall in the Ising model. He presented a convincing argument that *the behavior*

¹It is interesting to note that a similarity of the 't Hooft loop in LGT to the domain wall in spin models gets even clearer in the *deconfined* phase at high temperature. The action of $SU(N)$ LGT possesses a *global* Z_N symmetry, and the confinement-deconfinement transition is conventionally interpreted as its spontaneous breaking [8]. The tension of an interface separating different Z_N deconfined vacua is calculated perturbatively and numerically from the (dual) string tension of the spatial 't Hooft loop [9].

of the free energy of a large vortex in approaching the thermodynamic limit characterizes in which phase the system is in; if it vanishes exponentially, then the vortices percolates and the system is in the confining phase. On the other hand Mack and Petkova formulated a center vortex contained in a torus of finite diameter with a fixed boundary condition on the surface, and the presence of a center vortex was ensured by a singular gauge transformation operated on the surface. In order to elucidate its intuitive meaning to spin theorists, we would like comment on the concept of ‘thickness’ of the vortex. It is well known that, in the continuum, an *infinitely thin* center vortex is unphysical in the sense that it is associated with an infinite action. For illustration let us consider the XY model on a one-dimensional chain of length L . Suppose we fix the angle at one end of the chain to φ and the angle at the other end to $\varphi + \theta$. If the angles of spins change smoothly as much as possible from one end toward the other, the energy cost is easily estimated to be $(\theta/L)^2 \times L \sim O(1/L)$ for $L \gg 1$. This is in sharp contrast to the situation in the Ising model on the same chain, in which a smooth change is impossible, thus leading to the energy cost of $O(1)$ and making a spontaneous symmetry breaking easier to happen. The lesson we learn in this example is that it is generally possible to reduce an energy cost associated with a defect by smoothly changing the variables around it; the energy cost associated with a domain wall can be reduced if we give it a finite thickness.² What Mack and Petkova achieved is to prove an inequality rigorously, whose intuitive interpretation being that *the area-law decay of the Wilson loop expectation value would follow if the free energy of such a ‘thick’ vortex decreases sufficiently rapidly when its diameter is increased*. A lucid exposition of dynamics of thick vortices in $SU(N)$ lattice gauge theories (LGT) can be found in ref.[11].

As is well known, a fundamental difference between spin systems and gauge theories is that the latter has no known *local* order parameter (such as magnetization in the former) that can characterize the phases of gauge theories. That is why non-local quantities such as Wilson or ’t Hooft loops have been given a special weight in studies of strong-coupling phenomena such as confinement. As a classical reference, we would like to mention ref.[12] in which physical relevance of defects generated by the twisting procedure, including both center vortices in LGT and domain walls in spin systems, and usefulness of using them as a probe for the phase structure in computer simulations, are reviewed and discussed from a unified point of view.

Tomboulis and Yaffe thoroughly investigated $SU(2)$ LGT at finite temperature and rigorously proved the absence of confinement at sufficiently high temperature [1]. In their study they derived a number of inequalities between observables such as the Wilson loop expectation value, the ’t Hooft loop expectation value, the electric flux expectation value and the Polyakov loop correlator. Among others they gave an upper bound of the Wilson loop expectation value by a specific function of the center vortex free energy (as presented in the next section as theorem 1). It gave a firm foundation to ’t Hooft’s original argument in the continuum [5], that if in approaching the thermodynamic limit the free energy of a center vortex that encircles two of the four periodic directions of the lattice vanishes exponentially w.r.t. the cross section of the lattice perpendicular to the vortex, then the area law behavior of the Wilson loop expectation value would follow. Thus it sheds light on dynamics of the center vortices in a somewhat different manner from the Mack-Petkova inequality. In this paper we call it the Tomboulis-Yaffe (TY) inequality throughout this paper.

The purpose of this paper is to present a generalization of the TY inequality to $SU(N)$ LGT for arbitrary N and to general classical spin models. Our result gives a rigorous relation between the effect of twisted boundary conditions and the correlation function (Wilson loop) in spin models (in LGT) respectively.³ Among spin models, the $SU(N) \times SU(N)$ principal chiral model (PCM) is of

² Dobrushin and Shlosman elevated this idea to a rigorous proof of the absence of magnetic order in two-dimensional ferromagnets with a continuous symmetry [10].

³Historically, changing of boundary conditions has been utilized in studies of Anderson localization as a method for estimating the broadening of the wave function [13]. More recently it was utilized in the lattice QCD calculation [14]

particular interest for researches of gauge theory, since it bears a number of similarities to $SU(N)$ gauge theories and serves as a good testing ground for techniques in gauge theories [15, 16, 17, 18]. The action of $SU(N)$ PCM is given by

$$S = \beta \sum_x \sum_{\mu=1}^d \text{Re Tr} \{U(x)U^\dagger(x + \hat{\mu})\}, \quad U \in SU(N), \quad x \in \mathbb{Z}^d, \quad (1)$$

where $\hat{\mu}$ denotes a unit vector in x^μ -direction. It is quite straightforward to extend the original TY inequality for $SU(N)$ LGT to $SU(N)$ PCM, using a natural correspondence (site \leftrightarrow link, link \leftrightarrow plaquette, ...) and indeed the TY inequality for $SU(2)$ PCM has already appeared in the literature [19, 20]. On the other hand, however, it is technically nontrivial how to extend it to other more general spin models. Let us take G_2 PCM as an example. Since G_2 is an exceptional group with *trivial* center, we can no longer use a twist by a center of the gauge group, which gives rise to a technical difficulty. Furthermore the use of center twist for PCM is not physically motivated; in the case of $SU(N)$ gauge theory, the use of center element is mandatory, but in PCM we can use any other element of the symmetry group for twist. Thus the generality of our formulation, that does not rely on the center of the symmetry group at all, seems to be a fundamental progress.⁴

This paper is organized as follows. In section 2 we will recapitulate the TY inequality for $SU(2)$ and then prove its generalization to $SU(N)$. We will use the two-dimensional $SU(N)$ LGT to illustrate our result. In section 3 we will prove a generalization of the inequality to general classical spin systems. We will use the one-dimensional PCM and the two-dimensional Ising models on square and triangular lattices to illustrate the proved inequality. Especially, in section 3.5, we derive a rigorous upper bound of the off-axis correlation length in the triangular Ising model, whose exact expression is still unknown, and conjecture that it is indeed the exact one. In section 3.6 the strong coupling expansion technique is employed to shed light on the implication of our theorem, as well as to test the conjecture. Section 4 is devoted to the conclusion.

2 TY inequality in LGT

2.1 $N = 2$

Let us recapitulate the TY inequality for $SU(2)$ [1]. Λ is a d -dimensional hypercubic lattice of length L_μ ($\mu = 1, \dots, d$) with periodic boundary condition and \mathcal{V} , called “vortex”, is a stacked set of plaquettes winding around the lattice Λ in $d - 2$ periodic directions.⁵ We assume the directions unwrapped by \mathcal{V} to be x^μ and x^ν ($\mu \neq \nu$). The ordinary and the “twisted” partition functions are given by

$$Z_\Lambda \equiv \int \prod_b dU_b \exp \left(\frac{\beta}{2} \sum_{p \in \Lambda} \text{Tr} U_p \right), \quad (2)$$

$$Z_\Lambda^{(-)} \equiv \int \prod_b dU_b \exp \left(\frac{\beta}{2} \left[\sum_{p \in \mathcal{V}} \text{Tr} (-U_p) + \sum_{p \in \Lambda \setminus \mathcal{V}} \text{Tr} U_p \right] \right), \quad (3)$$

where dU is the normalized Haar measure of $SU(2)$ and $U_p \equiv U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^\dagger U_{x,\nu}^\dagger$ is a plaquette variable. It is important that local redefinition of variables $U \rightarrow -U$ can move the locations of twisted plaquettes but cannot remove the twist from Λ entirely.

to study charmonium properties in deconfinement phase.

⁴ As an aside we note that the inequality of Mack and Petkova for $SU(N)$ LGT was generalized to $SU(N)$ PCM by Borisenko and Skala [21].

⁵ \mathcal{V} forms a closed loop when $d = 3$ and a closed surface (2-torus) when $d = 4$, on the dual lattice. See fig.1.

Next, consider a rectangle C lying in a x^μ - x^ν plane with A_C the area enclosed by C , and let $W(C)$ the Wilson loop in the fundamental representation associated with C , namely $W(C) \equiv \frac{1}{2} \text{Tr} \prod_{b \in C} U_b$. Then the following inequality holds [1, 22]:

Theorem 1.

$$\langle W(C) \rangle \leq 2 \left\{ \frac{1}{2} \left(1 - \frac{Z_\Lambda^{(-)}}{Z_\Lambda} \right) \right\}^{A_C / L_\mu L_\nu} \quad (4)$$

where $\langle \dots \rangle$ is the expectation value w.r.t. the measure of Z_Λ .

The site-reflection positivity of the Wilson action [23] plays an essential role in the proof. (As is well known, the Wilson action is among those actions for which the link-reflection positivity is also satisfied [24] but it is not a matter of interest here.) Indeed (4) can be proved with any one-plaquette action, since they are site-reflection positive (although not necessarily link-reflection positive, of course).

An important implication of (4) is that *the area-law decay of $\langle W(C) \rangle$ would follow if $1 - Z_\Lambda^{(-)} / Z_\Lambda \approx e^{-\rho L_\mu L_\nu}$ for some constant $\rho > 0$ in the thermodynamic limit*⁶. This is a famous criterion of confinement originally proposed by 't Hooft [5] and is also numerically supported [25, 26, 27]. Moreover such a behavior of $Z_\Lambda^{(-)} / Z_\Lambda$ has been verified explicitly by Münster using the convergent strong-coupling cluster expansion technique [28]. See page 10 for more discussion on this point.

Theorem 1 was utilized in a recent attempt at a rigorous proof of confinement [29] with related discussions [30, 31].

2.2 General N

The authors of ref.[1] state without explicit construction that their result is extendable to any other gauge group with nontrivial center. Since the mentioned extension does not seem to be so trivial and, to the author's best knowledge an explicit formula for general N is not found in the literature, we think it valuable to present the extension of (4) from $SU(2)$ to $SU(N)$ together with its proof.

Let us give a formulation of vortices in $SU(N)$ LGT and prove their properties before presenting TY inequality in $SU(N)$ LGT. The ordinary and the “twisted” partition functions are respectively given by

$$Z_\Lambda \equiv \int \prod_b dU_b \exp \left(\frac{\beta}{N} \sum_{p \in \Lambda} \text{Re Tr } U_p \right), \quad (5)$$

$$Z_\Lambda^{[k]} \equiv \int \prod_b dU_b \exp \left(\frac{\beta}{N} \left[\sum_{p \in \mathcal{V}} \text{Re Tr } (z^k U_p) + \sum_{p \in \Lambda \setminus \mathcal{V}} \text{Re Tr } U_p \right] \right), \quad (6)$$

$$z \equiv \exp \left(\frac{2\pi i}{N} \right), \quad k \equiv 1, 2, \dots, N-1 \pmod{N}. \quad (7)$$

Hereafter $\langle \dots \rangle$ represents the expectation value with the measure (5). The vortex creation operators $\{\mathcal{O}^{[k]}\}$ and the electric flux creation operators $\{\mathcal{F}^{[m]}\}$ are defined by

$$\langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle \equiv \frac{Z_\Lambda^{[k]}}{Z_\Lambda}, \quad (8)$$

$$\langle \mathcal{F}^{[m]}[\mathcal{V}] \rangle \equiv \frac{1}{N} \sum_{k=0}^{N-1} z^{mk} \langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle. \quad (9)$$

⁶We neglected the entropy factor for simplicity.

Thus we have $\langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle = \sum_{m=0}^{N-1} z^{-km} \langle \mathcal{F}^{[m]}[\mathcal{V}] \rangle$. The explicit form of \mathcal{O} is given by

$$\mathcal{O}^{[k]}[\mathcal{V}] = \exp\left(\frac{\beta}{N} \sum_{p \in \mathcal{V}} [\text{Re Tr}(z^k U_p) - \text{Re Tr } U_p]\right). \quad (10)$$

Lemma 1. *If $\mathcal{V}, \mathcal{V}', \mathcal{V}'', \dots$ are homologous⁷, we have*

$$\langle \mathcal{O}^{[k]}[\mathcal{V}] \mathcal{O}^{[k']}[\mathcal{V}'] \rangle = \langle \mathcal{O}^{[k+k']}[\mathcal{V}] \rangle, \quad (11)$$

$$\sum_{m=0}^{N-1} \mathcal{F}^{[m]}[\mathcal{V}] = 1, \quad (12)$$

$$\langle \mathcal{F}^{[l]}[\mathcal{V}] \mathcal{F}^{[m]}[\mathcal{V}'] \rangle = \langle \mathcal{F}^{[l]}[\mathcal{V}] \rangle \delta_{l,m}^{(N)}, \quad (13)$$

$$\langle \mathcal{F}^{[l]}[\mathcal{V}] \mathcal{F}^{[m]}[\mathcal{V}'] \mathcal{F}^{[n]}[\mathcal{V}'] \rangle = \langle \mathcal{F}^{[l]}[\mathcal{V}] \rangle \delta_{l,m}^{(N)} \delta_{m,n}^{(N)}, \quad (14)$$

\vdots

where $\delta_{l,m}^{(N)} = 1$ if $l \equiv m \pmod{N}$ and $\delta_{l,m}^{(N)} = 0$ otherwise.

Proof. (11) can be derived by iterating the redefinition of variables $U \rightarrow z^{k'} U$ to bring \mathcal{V}' to \mathcal{V} . Relations (12)-(14) follow from (9) and (11). \square

(12)-(14) imply that $\{\mathcal{F}^{[m]}\}_m$ can be seen as projection operators [5, 11].

Consider a $(d-1)$ -dimensional hyperplane π defined by $x^\mu = m$ with $m \in \mathbb{Z}$ fixed.⁸ Define the reflection operator θ w.r.t. π by $\theta[F(\{U_b\})] = \overline{F(\{U_{\theta[b]}\})}$ where F is an arbitrary observable (that is, a map from configurations on Λ to \mathbb{C}). The reflection $\theta[b]$ of a link b is also defined by the same notation where locations of b and $\theta[b]$, are defined to be symmetrical about π .

Lemma 2. *With $\mathcal{V}^\theta \equiv \theta[\mathcal{V}]$ we have*

$$\theta[\mathcal{O}^{[k]}[\mathcal{V}]] = \mathcal{O}^{[-k]}[\mathcal{V}^\theta], \quad (15)$$

$$\theta[\mathcal{F}^{[m]}[\mathcal{V}]] = \mathcal{F}^{[m]}[\mathcal{V}^\theta], \quad (16)$$

$$0 \leq \langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle \leq 1, \quad (17)$$

$$0 \leq \langle \mathcal{F}^{[m]}[\mathcal{V}] \rangle \leq 1. \quad (18)$$

Proof. (15) is obvious from the fact that the orientation of plaquettes are reversed by reflection. (15) yields

$$\theta[\mathcal{F}^{[m]}[\mathcal{V}]] = \theta\left[\frac{1}{N} \sum_{k=0}^{N-1} z^{mk} \mathcal{O}^{[k]}[\mathcal{V}]\right] \quad (19)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} z^{-mk} \mathcal{O}^{[-k]}[\mathcal{V}^\theta] \quad (20)$$

$$= \mathcal{F}^{[m]}[\mathcal{V}^\theta] \quad (21)$$

which proves (16). Next, using the Schwarz inequality $|\langle F \rangle| \leq \langle F \theta F \rangle^{1/2}$ and (11), (15) we find

$$\langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle \leq \langle \mathcal{O}^{[k]}[\mathcal{V}] \theta[\mathcal{O}^{[k]}[\mathcal{V}]] \rangle^{1/2} \quad (22)$$

$$= \langle \mathcal{O}^{[k]}[\mathcal{V}] \mathcal{O}^{[-k]}[\mathcal{V}^\theta] \rangle^{1/2} = 1. \quad (23)$$

⁷Plural vortices are called *homologous* if and only if they wind around the same periodic directions of Λ .

⁸ In this paper we never use hyperplanes defined by $x^\mu = m + \frac{1}{2}$; that is, we never use link-reflections.

which proves the second inequality in (17) while the first one is trivial. Since \mathcal{V} and \mathcal{V}^θ are homologous we can apply (13) to obtain

$$\langle \mathcal{F}^{[m]}[\mathcal{V}] \rangle = \langle \mathcal{F}^{[m]}[\mathcal{V}] \mathcal{F}^{[m]}[\mathcal{V}^\theta] \rangle \quad (24)$$

$$= \langle \mathcal{F}^{[m]}[\mathcal{V}] \theta \left[\mathcal{F}^{[m]}[\mathcal{V}] \right] \rangle \geq 0. \quad (25)$$

(25) and (12) prove (18). (These simple proofs of (17) and (18) seem to be new.) \square

The vortex free energy $F_v^{[k]}$ and the electric flux free energy $F_{el}^{[m]}$ are defined by $e^{-F_v^{[k]}} \equiv \langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle$ and $e^{-F_{el}^{[m]}} \equiv \langle \mathcal{F}^{[m]}[\mathcal{V}] \rangle$, respectively.

Let $N(R) \in \{0, 1, \dots, N-1\}$ denote the N -ality of an irreducible representation R^9 of $SU(N)$ whose dimension is d_R . Take a rectangle C lying in a x^μ - x^ν plane and let A_C the area enclosed by C . For the normalized Wilson loop in the representation R , $W_R(C) \equiv \frac{1}{d_R} \chi_R \left(\prod_{b \in C} U_b \right)$, we have

Theorem 2 (TY inequality for $SU(N)$ LGT).

$$|\langle W_R(C) \rangle| \leq \langle \mathcal{F}^{[N(R)]}[\mathcal{V}] \rangle^{A_C/L_\mu L_\nu} + \left\{ 1 - \langle \mathcal{F}^{[0]}[\mathcal{V}] \rangle \right\}^{A_C/L_\mu L_\nu}. \quad (26)$$

In addition, if $N(R) \neq 0$ we have

$$|\langle W_R(C) \rangle| \leq 2 \left\{ 1 - \langle \mathcal{F}^{[0]}[\mathcal{V}] \rangle \right\}^{A_C/L_\mu L_\nu} \quad (27)$$

$$= 2 \left\{ 1 - \frac{1}{N} \sum_{k=0}^{N-1} \langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle \right\}^{A_C/L_\mu L_\nu}. \quad (28)$$

Proof. Although the argument below parallels that of ref.[22] for $SU(2)$, we describe the proof in full detail for readers' convenience. Suppose \mathcal{V} , \mathcal{V}' are stacked set of plaquettes wrapping around $d-2$ periodic directions of Λ and \mathcal{V} is linking once with C while \mathcal{V}' is not. (See fig.1 for a 3-dimensional illustration of the setting.)

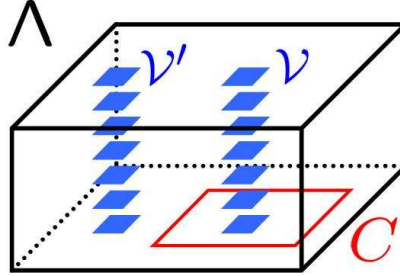


Figure 1: Locations of \mathcal{V} , \mathcal{V}' and C .

Let us rewrite the expectation value of $W_R(C) \equiv \frac{1}{d_R} \chi_R \left(\prod_{b \in C} U_b \right)$ as follows:

$$\langle W_R(C) \rangle = \langle (1 - \mathcal{F}^{[0]}[\mathcal{V}]) W_R(C) \rangle + \langle \mathcal{F}^{[0]}[\mathcal{V}] W_R(C) \rangle \quad (29)$$

$$= \langle (1 - \mathcal{F}^{[0]}[\mathcal{V}]) W_R(C) \rangle + \frac{1}{N} \sum_{k=0}^{N-1} \langle \mathcal{O}^{[k]}[\mathcal{V}] W_R(C) \rangle. \quad (30)$$

⁹ N -ality is the number (mod N) of boxes in the Young tableau of R .

The presence of the second term is not desirable from the viewpoint of obtaining a meaningful upper bound of $\langle W_R(C) \rangle$, so let us perform redefinitions of variables $U \rightarrow z^k U$ to bring \mathcal{V} to \mathcal{V}' , which causes the change

$$\langle \mathcal{O}^{[k]}[\mathcal{V}] W_R(C) \rangle \rightarrow z^{\pm N(R)k} \langle \mathcal{O}^{[k]}[\mathcal{V}'] W_R(C) \rangle. \quad (31)$$

This is because in the course of bringing \mathcal{V} to \mathcal{V}' we must change one of the link variables on C . (The sign of exponent depends on the orientation of C .) Thus (30) becomes

$$\langle W_R(C) \rangle = \langle (1 - \mathcal{F}^{[0]}[\mathcal{V}]) W_R(C) \rangle + \frac{1}{N} \sum_{k=0}^{N-1} z^{\pm N(R)k} \langle \mathcal{O}^{[k]}[\mathcal{V}'] W_R(C) \rangle \quad (32)$$

$$= \langle (1 - \mathcal{F}^{[0]}[\mathcal{V}]) W_R(C) \rangle + \langle \mathcal{F}^{[\pm N(R)]}[\mathcal{V}'] W_R(C) \rangle. \quad (33)$$

Our next step is expressed in fig.2 schematically in which a black square represents the oper-

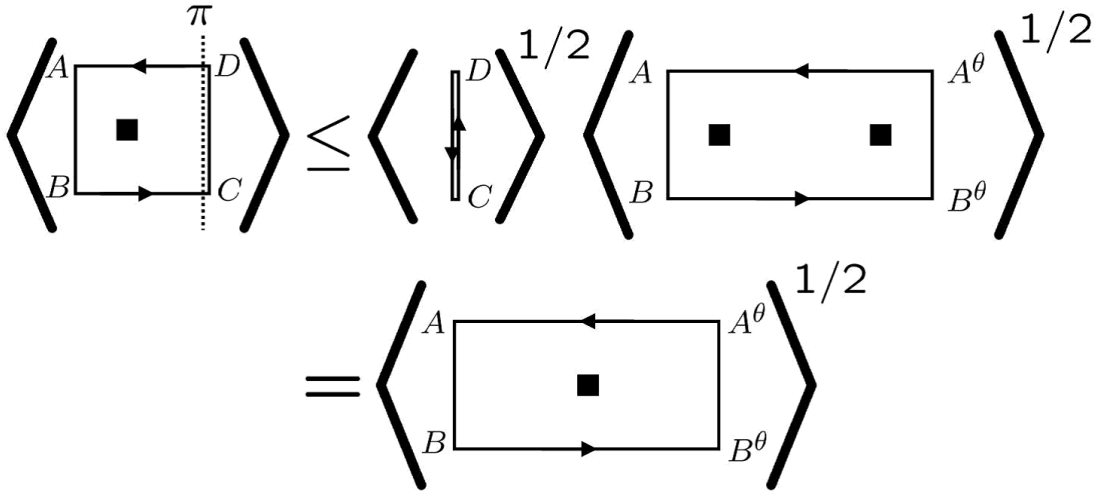


Figure 2: The Schwarz inequality enables us to double the size of the rectangle. This figure is essentially borrowed from ref.[22].

ator $(1 - \mathcal{F}^{[0]}[\mathcal{V}])$. Labeling four vertices as A, B, C, D , setting the hyperplane π so that it is perpendicular to the rectangle and contains the edge CD , and applying the Schwarz inequality

$|\langle F\theta G \rangle| \leq \langle F\theta F \rangle^{1/2} \langle G\theta G \rangle^{1/2}$ we obtain

$$|\langle (1 - \mathcal{F}^{[0]}[\mathcal{V}])W_R|_{DA+AB+BC+CD} \rangle| \quad (34)$$

$$= \frac{1}{d_R} |\langle (1 - \mathcal{F}^{[0]}[\mathcal{V}])\chi_R|_{DA+AB+BC+CD} \rangle| \quad (35)$$

$$= \frac{1}{d_R} \left| \sum_{\alpha, \beta} \langle (1 - \mathcal{F}^{[0]}[\mathcal{V}])\{\chi_R|_{DA+AB+BC}\}_{\alpha\beta} \{\chi_R|_{CD}\}_{\beta\alpha} \rangle \right| \quad (36)$$

$$\leq \frac{1}{d_R} \sum_{\alpha, \beta} \left\langle (1 - \mathcal{F}^{[0]}[\mathcal{V}]) (1 - \mathcal{F}^{[0]}[\mathcal{V}^\theta]) \{\chi_R|_{DA+AB+BC}\}_{\alpha\beta} \overline{\{\chi_R|_{DA^\theta+A^\theta B^\theta+B^\theta C}\}_{\alpha\beta}} \right\rangle^{1/2} \\ \times \left\langle \{\chi_R|_{CD}\}_{\beta\alpha} \overline{\{\chi_R|_{CD}\}_{\beta\alpha}} \right\rangle^{1/2} \quad (37)$$

$$\leq \frac{1}{d_R} \left\langle (1 - \mathcal{F}^{[0]}[\mathcal{V}]) \sum_{\alpha, \beta} \{\chi_R|_{DA+AB+BC}\}_{\alpha\beta} \overline{\{\chi_R|_{DA^\theta+A^\theta B^\theta+B^\theta C}\}_{\alpha\beta}} \right\rangle^{1/2} \times \sqrt{d_R} \quad (38)$$

$$= \langle (1 - \mathcal{F}^{[0]}[\mathcal{V}])W_R|_{AB+BB^\theta+B^\theta A^\theta+A^\theta A} \rangle^{1/2}. \quad (39)$$

In going from (37) to (38) we used (13). The length of the rectangle doubled.

Let $l_\mu \times l_\nu$ the size of the original rectangle and assume that $L_\mu = 2^p l_\mu$ and $L_\nu = 2^q l_\nu$ for some $p, q \in \mathbf{N}^{10}$. Repeating the operation above for sufficiently many times in both x^μ - and x^ν - directions, due to the periodic boundary conditions the rectangle finally vanishes away, yielding

$$\langle (1 - \mathcal{F}^{[0]}[\mathcal{V}])W_R(C) \rangle \leq \left\{ 1 - \langle \mathcal{F}^{[0]}[\mathcal{V}] \rangle \right\}^{1/2^{p+q}} \quad (40)$$

$$= \left\{ 1 - \langle \mathcal{F}^{[0]}[\mathcal{V}] \rangle \right\}^{l_\mu l_\nu / L_\mu L_\nu} = \left\{ 1 - \langle \mathcal{F}^{[0]}[\mathcal{V}] \rangle \right\}^{A_C / L_\mu L_\nu}. \quad (41)$$

$\langle \mathcal{F}^{[\pm N(R)]}[\mathcal{V}']W_R(C) \rangle \leq \langle \mathcal{F}^{[\pm N(R)]}[\mathcal{V}'] \rangle^{A_C / L_\mu L_\nu}$ can be shown in a similar way, hence (26) is proved.

(27) is a consequence of (26) and $\langle \mathcal{F}^{[N(R)]}[\mathcal{V}] \rangle \leq 1 - \langle \mathcal{F}^{[0]}[\mathcal{V}] \rangle$. \square

The message of (28) is that *the exponential decay of the vortex free energy, i.e. $\langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle \equiv e^{-F_v^{[k]}} = 1 - O(e^{-\rho L_\mu L_\nu})$ for every k , is a sufficient condition for the area law of the Wilson loop to hold.* The area law does not hold, or is at least difficult to prove, if not all of the $\langle \mathcal{O} \rangle$'s converge to 1.

Several comments are in order. Firstly, suppose that the action in (2) is in the *adjoint* representation. Then $\langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle = 1$ follows for any k , hence making $\langle W_R(C) \rangle$ with $N(R) \neq 0$ vanish identically for arbitrary finite volume (see (28)). This is to be anticipated; since the adjoint Wilson action is invariant under the *local* transformation $U \rightarrow zU$ ($z \in Z_N$), which cannot break spontaneously according to the Elitzur's theorem, and since $W_R(C)$ with $N(R) \neq 0$ transforms nontrivially under this transformation, its expectation value *must* vanish.

Secondly, there are *a lot more varieties* of inequalities available other than (26). Assume $N = 6$ for instance. From (13) we have $\langle (\mathcal{F}^{[1]}[\mathcal{V}] + \mathcal{F}^{[2]}[\mathcal{V}]) (\mathcal{F}^{[1]}[\mathcal{V}'] + \mathcal{F}^{[2]}[\mathcal{V}']) \rangle = \langle \mathcal{F}^{[1]}[\mathcal{V}] + \mathcal{F}^{[2]}[\mathcal{V}] \rangle$ and $\langle (\mathcal{F}^{[3]}[\mathcal{V}] + \mathcal{F}^{[4]}[\mathcal{V}] + \mathcal{F}^{[5]}[\mathcal{V}]) (\mathcal{F}^{[3]}[\mathcal{V}'] + \mathcal{F}^{[4]}[\mathcal{V}'] + \mathcal{F}^{[5]}[\mathcal{V}']) \rangle = \langle \mathcal{F}^{[3]}[\mathcal{V}] + \mathcal{F}^{[4]}[\mathcal{V}] + \mathcal{F}^{[5]}[\mathcal{V}] \rangle$, hence by modifying the above proof one can straightforwardly show

$$|\langle W_R(C) \rangle| \leq \langle \mathcal{F}^{[N(R)]} \rangle^{A_C / L_\mu L_\nu} + \left\{ \langle \mathcal{F}^{[1]} + \mathcal{F}^{[2]} \rangle \right\}^{A_C / L_\mu L_\nu} + \left\{ \langle \mathcal{F}^{[3]} + \mathcal{F}^{[4]} + \mathcal{F}^{[5]} \rangle \right\}^{A_C / L_\mu L_\nu}. \quad (42)$$

However (42) and all of its cousins are weaker than (26) with $N = 6$, which can be understood by the elementary inequality $(\sum_i x_i)^\alpha < \sum_i (x_i)^\alpha$ for $0 < \alpha < 1$ and $0 < x_i$.

¹⁰This condition was also present in the original TY inequality [1]. It is not a severe restriction, however, as long as we believe the asymptotic behavior of observables to be independent of the way we take L_μ, L_ν to infinity.

Note that one cannot derive the area law from (26) when $N(R) = 0$, as can be seen from

$$\left[\text{r.h.s. of (26)} \right] \geq \langle \mathcal{F}^{[0]}[\mathcal{V}] \rangle^{A_C/L_\mu L_\nu} \geq \left(\frac{1}{N} \right)^{A_C/L_\mu L_\nu} \rightarrow 1 \text{ as } L_\mu, L_\nu \rightarrow \infty. \quad (43)$$

The above implies that the “gluons” of $SU(N)$ can screen particles of zero N -ality.

Thirdly, theorem 2 is correct even after a matter field whose N -ality is zero is introduced into the theory, since the matter-gauge coupling $\Phi_{x+\mu}^\dagger D_r[U_{x,\mu}] \Phi_x$ preserves reflection positivity and is insensitive to the change of variables $U \rightarrow zU$.

Finally we remark on the utility of strong-coupling cluster expansion techniques. (Similar discussion will be presented in section 3.6.) As already mentioned, the exponential suppression of vortex free energy $-\log\langle \mathcal{O}[\mathcal{V}] \rangle \approx e^{-\rho L_\mu L_\nu}$ has been verified by Münster [28] for $SU(2)$ LGT and for sufficiently strong coupling. Especially he showed to all orders of strong-coupling expansion that the constant ρ appearing in the vortex free energy (’t Hooft’s string tension) is equal to the conventional Wilson’s string tension. His proof hinges on the observation that both the calculation of Wilson loop expectation value and that of vortex free energy reduce, at sufficiently strong coupling, to the problem of fluctuating *random surfaces*. It is understood without difficulty that the methods he employed can be readily used for $SU(N)$ LGT to show $-\log\langle \mathcal{O}^{[k]}[\mathcal{V}] \rangle \approx e^{-\rho L_\mu L_\nu}$ for every $k \neq 0$ (hence proving the area law). This time ρ is equal to the *fundamental* string tension (since the gauge action (5) is in the fundamental representation).

* * * *

Let us then turn to LGT with matter field of *non-zero* N -ality; the relations (11)-(14) are no longer valid. If the matter field has N -ality m and the greatest common divisor of N and m is s , the subgroup $Z_s \subset Z_N$ is a symmetry of the theory. It is thus straightforward to prove the following

Theorem 3. *If $N(R) \not\equiv 0 \pmod{s}$, we have*

$$|\langle W_R(C) \rangle| \leq 2 \left\{ 1 - \langle f^{[0]}[\mathcal{V}] \rangle \right\}^{A_C/L_\mu L_\nu} = 2 \left\{ 1 - \frac{1}{s} \sum_{k=0}^{s-1} \frac{Z_\Lambda^{[kN/s]}}{Z_\Lambda} \right\}^{A_C/L_\mu L_\nu}. \quad (44)$$

Although the development so far has been for $SU(N)$ LGT, the inequalities evidently apply to $U(N)$ LGT since the center of $U(N)$ is $U(1)$ which contains all of $Z_2, Z_3, Z_4, Z_5, \dots$. Let us focus on $U(1)$ for simplicity and define the twisted partition function as

$$Z_\Lambda(\theta) \equiv \int \prod_b dU_b \exp \left(\beta \left[\sum_{p \in \mathcal{V}} \text{Re}(e^{i\theta} U_p) + \sum_{p \in \Lambda \setminus \mathcal{V}} \text{Re} U_p \right] \right). \quad (45)$$

We state below the counterpart of theorem 2. The proof is straightforward.

Theorem 4. *For the Wilson loop of $U(1)$ -charge $q \in \mathbb{Z} \setminus \{0\}$, we have*

$$|\langle W_q(C) \rangle| \leq 2 \left\{ 1 - \langle \mathcal{F}_{U(1)}^{[0]}[\mathcal{V}] \rangle \right\}^{A_C/L_\mu L_\nu} = 2 \left\{ \int_0^{2\pi} \frac{d\theta}{2\pi} \left(1 - \frac{Z_\Lambda(\theta)}{Z_\Lambda} \right) \right\}^{A_C/L_\mu L_\nu}, \quad (46)$$

with

$$\langle \mathcal{F}_{U(1)}^{[0]}[\mathcal{V}] \rangle \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{Z_\Lambda(\theta)}{Z_\Lambda}. \quad (47)$$

Finally we point out that theorem 2 can be proved even if we add a Wilson loop with **zero** N -ality, of size 2×2 or 2×1 to the action. It is simply because the site-reflection positivity is kept and the algebras of twists are still well defined.

2.3 Demonstration in 2D $SU(N)$ LGT

Let us explicitly verify the proved inequality in solvable two-dimensional $SU(N)$ LGT. In two dimension, the twist is introduced on just one plaquette. Let the size of the lattice $L_1 \times L_2$ and impose periodic boundary conditions in both directions. First we expand the exponentiated one-plaquette action, e^{-S_p} , into characters of irreducible unitary representations of $SU(N)$:

$$e^{-S_p(U)} = \sum_r d_r F_r \chi_r(U) \quad (48)$$

where d_r denotes the dimension of a representation r and the reality of S_p implies $F_r = F_{\bar{r}}$ (overline represents complex conjugation). Then a straightforward calculation using formulae

$$\int dU \chi_r(VU) \chi_{r'}(U^\dagger W) = \frac{1}{d_r} \delta_{rr'} \chi_r(VW), \quad (49)$$

$$\int dU \chi_r(VUWU^\dagger) = \frac{1}{d_r} \chi_r(V) \chi_r(W), \quad (50)$$

yields¹¹

$$Z_\Lambda \equiv \int \prod_{b \in \Lambda} dU_b \prod_p \left(\sum_r d_r F_r \chi_r(U_p) \right) \quad (51)$$

$$= \sum_r (F_r)^{L_1 L_2}. \quad (52)$$

On the other hand, introducing a twist $z^k = \exp(2\pi i k/N)$ into arbitrary one plaquette on Λ gives the twisted partition function

$$Z_\Lambda^{[k]} = \sum_r (F_r)^{L_1 L_2} z^{kN(r)}. \quad (53)$$

Using the identity $\sum_{k=0}^{N-1} z^{kN(r)} = N \delta_{0, N(r)}$ we easily obtain

$$1 - \frac{1}{N} \sum_{k=0}^{N-1} \langle \mathcal{O}^{[k]} \rangle = \frac{\sum_{r; N(r) \neq 0} (F_r)^{L_1 L_2}}{\sum_r (F_r)^{L_1 L_2}} \quad (54)$$

$$= \frac{\sum_{r; N(r) \neq 0} (c_r)^{L_1 L_2}}{1 + \sum_{r \neq T} (c_r)^{L_1 L_2}} \quad (55)$$

where T implies the trivial representation and we defined $c_r \equiv \frac{F_r}{F_T}$. Note that $c_r = c_{\bar{r}}$. Since

$$|F_r| = \left| \frac{1}{d_r} \int dU e^{-S_p(U)} \overline{\chi_r(U)} \right| < \int dU e^{-S_p(U)} = F_T, \quad (56)$$

¹¹(52) differs from that obtained in ref.[32] because they impose *free* boundary conditions.

we have $|c_r| < 1$, while $0 \leq c_r$ can be shown for a wide class of gauge actions including the Wilson action.

From above considerations we obtain

$$\left[\text{r.h.s. of (27)} \right] = 2 \left\{ 1 - \frac{1}{N} \sum_{k=0}^{N-1} \langle \mathcal{O}^{[k]} \rangle \right\}^{A_C / L_1 L_2} \quad (57)$$

$$\rightarrow 2(c_{r'})^{A_C} \quad \text{as } L_1 L_2 \rightarrow \infty. \quad (58)$$

Here $c_{r'}$ is defined as the largest value among $\{c_r \mid N(r) \neq 0\}$. In order to determine r' we need an explicit form of the action S_p .

Let us turn to the l.h.s. of (27), i.e. the Wilson loop expectation value. We borrow the result of ref.[32] which in our notation reads

$$\langle W_R(C) \rangle = (c_{\overline{R}})^{A_C}. \quad (59)$$

If $N(R) \neq 0$, we obviously have $c_{\overline{R}} \leq c_{r'}$, hence the inequality (27) holds for sure.

3 Extension of inequalities to spin systems

Main result of this section is theorem 5 on page 17, which is a generalization of theorem 2 to general spin systems. Before that, we need some preliminary analyses.

3.1 Basic formulation

Let us formulate a twisting procedure in spin systems obeying ref.[12]. Consider a statistical system with nearest-neighbor interactions whose partition function is given by

$$Z_\Lambda \equiv \int \prod_{x \in \Lambda} d\phi_x \exp \left(\sum_{y \in \Lambda} \sum_{\mu=1}^d A(\phi_y, \phi_{y+\hat{\mu}}) \right), \quad (60)$$

where Λ is a d -dimensional hypercubic lattice with periodic boundary conditions, $\hat{\mu}$ is a unit vector in the μ -direction and $\sum_{y \in \Lambda} \sum_{\mu=1}^d$ is a sum over all links in Λ . The real-valued symmetric function $A(,)$ dictates the interaction between nearest sites (and possibly includes self interactions on each site). It can be shown by standard arguments that site-reflection positivity is automatically satisfied for any nearest neighbor interaction (see p.33 of ref.[23]) while link-reflection, not needed in the following, is often violated. Hereafter $\langle \dots \rangle$ represents the expectation value with the measure (60). Let us assume that the system is invariant under a global transformation $\phi \rightarrow g\phi$ for any element g of a global symmetry group G :¹²

$$A(\phi, \phi') = A(g\phi, g\phi'), \quad g \in G. \quad (61)$$

We assume that G is compact.

A twist for a link is defined as the change of interaction from $A(\phi, \phi')$ to $A(\phi, g\phi')$. An important difference from the twist in LGT is that g need not belong to the center of G . G may or may not have a nontrivial center and that is not important for us.

Next, let us take a stacked set of links, \mathcal{V} , which winds around Λ in $d-1$ periodic directions (\mathcal{V} is a closed loop when $d=2$ and a closed surface when $d=3$ on the dual lattice, see fig. 3). Hereafter such \mathcal{V} is called a *wall* in distinction from a (center) vortex.

¹²Note that G need not be the maximal symmetry group of the system. The development in this section still holds if we take as G an arbitrary subgroup of the maximal symmetry group.

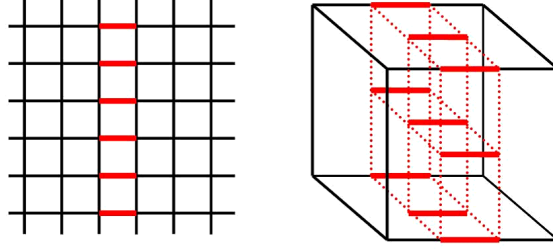


Figure 3: \mathcal{V} in two- and three- dimensions.

The twisted partition function associated to $g \in G$ is given by

$$Z_\Lambda(g)[\mathcal{V}] \equiv \int \prod_{x \in \Lambda} d\phi_x \exp \left(\sum_{y \in \Lambda} \sum_{\mu=1}^d \left[(1 - \delta[\mathcal{V}]_{y,\mu}) A(\phi_y, \phi_{y+\hat{\mu}}) + \delta[\mathcal{V}]_{y,\mu} A(\phi_y, g\phi_{y+\hat{\mu}}) \right] \right). \quad (62)$$

The symbol $\delta[\mathcal{V}]_{y,\mu}$ is defined to be $= 1$ if the link $\langle y, y+\hat{\mu} \rangle$ is contained in \mathcal{V} and $= 0$ otherwise. It is not difficult to see that \mathcal{V} cannot be removed from Λ by local redefinition of variables $\phi \rightarrow g\phi$. Let \hat{G} denote the set of irreducible unitary representations of G . The wall creation operators $\{\mathcal{O}(g)[\mathcal{V}] | g \in G\}$ and their duals $\{\mathcal{F}_R(g)[\mathcal{V}] | g \in G, R \in \hat{G}\}$ are defined as follows:

$$\langle \mathcal{O}(g)[\mathcal{V}] \rangle \equiv \frac{Z_\Lambda(g)[\mathcal{V}]}{Z_\Lambda}, \quad (63)$$

$$\iff \mathcal{O}(g)[\mathcal{V}] = \exp \left(\sum_{y \in \Lambda} \sum_{\mu=1}^d \delta[\mathcal{V}]_{y,\mu} \left[-A(\phi_y, \phi_{y+\hat{\mu}}) + A(\phi_y, g\phi_{y+\hat{\mu}}) \right] \right), \quad (64)$$

$$\mathcal{F}_R(g)[\mathcal{V}] \equiv (\dim R) \int_G dx \mathcal{O}(gx)[\mathcal{V}] \chi_R(x). \quad (65)$$

Lemma 3. *If the walls $\mathcal{V}, \mathcal{V}'$ are homologous¹³, we have*

$$\langle \mathcal{O}(g)[\mathcal{V}] \cdot \mathcal{O}(g')[\mathcal{V}'] \rangle = \langle \mathcal{O}(gg')[\mathcal{V}] \rangle = \langle \mathcal{O}(g'g)[\mathcal{V}] \rangle, \quad (66)$$

$$\langle \mathcal{F}_R(g)[\mathcal{V}] \cdot \mathcal{F}_{R'}(g')[\mathcal{V}'] \rangle = \delta_{RR'} \langle \mathcal{F}_R(gg')[\mathcal{V}] \rangle = \delta_{RR'} \langle \mathcal{F}_R(g'g)[\mathcal{V}] \rangle, \quad (67)$$

$$\langle \mathcal{O}(g)[\mathcal{V}] \rangle = \sum_{R \in \hat{G}} \langle \mathcal{F}_R(g)[\mathcal{V}] \rangle, \quad (68)$$

$$1 = \sum_{R \in \hat{G}} \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}] \rangle. \quad (69)$$

(66), (67) imply that $\langle \mathcal{O}(g)[\mathcal{V}] \rangle$ and $\langle \mathcal{F}_R(g)[\mathcal{V}] \rangle$ are class functions on G .

Proof. (66) is trivial, since the relative position of walls can be reversed owing to the periodic boundary

¹³Plural walls are called *homologous* if and only if they wind around the same periodic directions of Λ .

condition (see fig. 4). (67) can be shown by exploiting the invariance of the Haar measure:

$$\langle \mathcal{F}_R(g) \mathcal{F}_{R'}(g') \rangle = (\dim R)(\dim R') \left\langle \int_G dx \mathcal{O}(gx) \chi_R(x) \int_G dy \mathcal{O}(g'y) \chi_{R'}(y) \right\rangle \quad (70)$$

$$= (\dim R)(\dim R') \int_G dx \int_G dy \langle \mathcal{O}(g x g' y) \rangle \chi_R(x) \chi_{R'}(y) \quad (71)$$

$$= (\dim R)(\dim R') \int_G dx \langle \mathcal{O}(x) \rangle \int_G dy \chi_R((g g')^{-1} x y^{-1}) \chi_{R'}(y) \quad (72)$$

$$= \delta_{RR'} (\dim R) \int_G dx \langle \mathcal{O}(x) \rangle \chi_R((g g')^{-1} x) \quad (73)$$

$$= \delta_{RR'} \langle \mathcal{F}_R(g g') \rangle. \quad (74)$$

Finally, (68) is a consequence of the Peter-Weyl theorem [33] according to which any $f \in L^2(G)$ can be represented as

$$f(g) = \sum_{R \in \hat{G}} (\dim R) \int_G dx f(xg) \chi_R(x). \quad (75)$$

(69) trivially follows from (68). \square

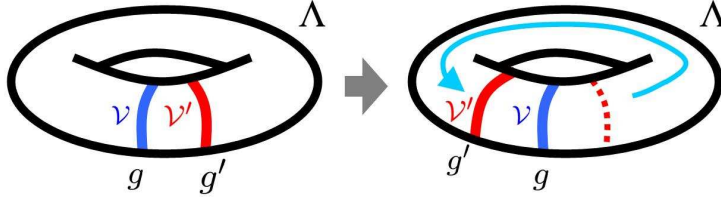


Figure 4: Relative position of \mathcal{V} and \mathcal{V}' can be reversed by using the periodicity of Λ .

Note that (66) cannot be proved in general when other operators are inserted, that is,

$$\langle \mathcal{O}(g)[\mathcal{V}] \cdot \mathcal{O}(g')[\mathcal{V}'] \dots \rangle \neq \langle \mathcal{O}(g g')[\mathcal{V}] \dots \rangle \neq \langle \mathcal{O}(g' g)[\mathcal{V}] \dots \rangle, \quad (76)$$

in general, where \dots denote additional insertions. It is because the proof of (66) involves a sequence of changes of variables. Thus (66) could be proved if inserted operators are invariant under the changes of variables.

Take an arbitrary $(d-1)$ -dimensional hyperplane π defined by $x^\mu = m$, $m \in \mathbb{Z}$ with m and μ fixed. Denote by θ the reflection about π .

Lemma 4. *If the wall \mathcal{V} seen on the dual lattice is also perpendicular to the x^μ -axis (see the left of fig. 5), we have*

$$\theta[\mathcal{O}(g)[\mathcal{V}]] = \mathcal{O}(g^{-1})[\mathcal{V}^\theta], \quad (77)$$

$$\langle \mathcal{O}(g)[\mathcal{V}] \rangle = \langle \mathcal{O}(g^{-1})[\mathcal{V}] \rangle, \quad (78)$$

$$\langle \mathcal{F}_R(g)[\mathcal{V}] \rangle = \overline{\langle \mathcal{F}_R(g^{-1})[\mathcal{V}] \rangle} \quad \text{and} \quad \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}] \rangle \in \mathbf{R}, \quad (79)$$

$$\theta[\mathcal{F}_R(g)[\mathcal{V}]] = \mathcal{F}_R(g^{-1})[\mathcal{V}^\theta], \quad (80)$$

where $\mathcal{V}^\theta \equiv \theta[\mathcal{V}]$ and $\mathbf{1}$ denotes the unit element of G .

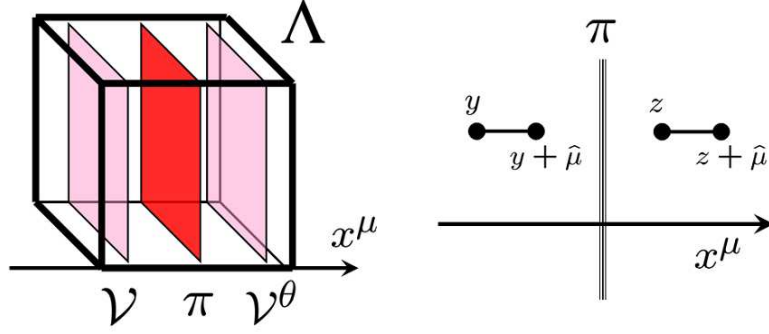


Figure 5: Insertion of a reflection plane π .

Proof. From the right of fig. 5 we observe

$$\theta \left[A(\phi_y, g\phi_{y+\hat{\mu}}) \right] = A(\phi_{z+\hat{\mu}}, g\phi_z) \quad (81)$$

$$= A(\phi_z, g^{-1}\phi_{z+\hat{\mu}}). \quad (82)$$

The twist g changed to g^{-1} , hence (77) is proved. (78) immediately follows from (77).

Using (78) we can show

$$\int_G dx \langle \mathcal{O}(gx)[\mathcal{V}] \rangle \chi_R(x) = \int_G dx \langle \mathcal{O}((gx)^{-1})[\mathcal{V}] \rangle \chi_R(x) \quad (83)$$

$$= \int_G dx \langle \mathcal{O}(x^{-1}g^{-1})[\mathcal{V}] \rangle \chi_R(x) \quad (84)$$

$$= \int_G dx \langle \mathcal{O}(xg^{-1})[\mathcal{V}] \rangle \chi_R(x^{-1}) \quad (85)$$

$$= \int_G dx \langle \mathcal{O}(g^{-1}x)[\mathcal{V}] \rangle \overline{\chi_R(x)}, \quad (86)$$

therefore (79) is proved. In the last step we used the fact that R is a unitary representation. (80) is obvious from (79). \square

Though it seems hard to find more properties on general grounds, further nontrivial result can be obtained if we exploit the site-reflection positivity of the measure of (60).

Lemma 5.

$$\langle \mathcal{O}(g)[\mathcal{V}] \rangle, \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}] \rangle \in [0, 1], \quad (87)$$

$$|\langle \mathcal{F}_R(g)[\mathcal{V}] \rangle| \leq \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}] \rangle. \quad (88)$$

Proof. Let $\mathcal{V}^\theta \equiv \theta[\mathcal{V}]$. With the aid of (66), (77) and the Schwarz inequality $|\langle F \rangle| \leq \langle F\theta F \rangle^{1/2}$ we get

$$\langle \mathcal{O}(g)[\mathcal{V}] \rangle \leq \langle \mathcal{O}(g)[\mathcal{V}] \theta[\mathcal{O}(g)[\mathcal{V}]] \rangle^{1/2} \quad (89)$$

$$= \langle \mathcal{O}(g)[\mathcal{V}] \mathcal{O}(g^{-1})[\mathcal{V}^\theta] \rangle^{1/2} = 1. \quad (90)$$

Next, using (67) and (80) yields

$$\langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}] \rangle = \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}] \cdot \mathcal{F}_R(\mathbf{1})[\mathcal{V}^\theta] \rangle \quad (91)$$

$$= \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}] \cdot \theta[\mathcal{F}_R(\mathbf{1})[\mathcal{V}]] \rangle \geq 0. \quad (92)$$

(92) combined with (69) yields (87).

Finally, to show (88) we use (80) and the Schwarz inequality $|\langle F\theta G \rangle| \leq \langle F\theta F \rangle^{1/2} \langle G\theta G \rangle^{1/2}$ as follows: letting \mathcal{V}' denote a wall homologous to \mathcal{V} , we have

$$|\langle \mathcal{F}_R(g)[\mathcal{V}] \rangle| = |\langle \mathcal{F}_R(g)[\mathcal{V}] \mathcal{F}_R(\mathbf{1})[\mathcal{V}'] \rangle| \quad (93)$$

$$\leq \langle \mathcal{F}_R(g)[\mathcal{V}] \mathcal{F}_R(g^{-1})[\mathcal{V}^\theta] \rangle^{1/2} \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}'] \mathcal{F}_R(\mathbf{1})[\mathcal{V}'^\theta] \rangle^{1/2} \quad (94)$$

$$= \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}] \rangle^{1/2} \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}'] \rangle^{1/2} \quad (95)$$

$$= \langle \mathcal{F}_R(\mathbf{1})[\mathcal{V}] \rangle. \quad (96)$$

□

3.2 TY inequality in spin systems

A natural counterpart in spin systems of Wilson loops in LGT is a two-point correlation function $\Gamma(\phi_x, \phi_y)$ as explained in the Introduction. Γ will decay exponentially with a mass gap (in symmetric phase) while decay algebraically without a mass gap (in a spontaneous symmetry breaking phase or Kosterlitz-Thouless-type phase). If $\langle \mathcal{O}(g) \rangle$ defined above converges to 1 in the thermodynamic limit, it follows that arbitrarily huge domain walls grow with little cost and eventually drive the system to the disordered phase with a mass gap (see fig. 6). $\langle \mathcal{O}(g) \rangle \rightarrow 1$ ($|\Lambda| \rightarrow \infty$) can also be regarded as a sign of insensitivity of the system to boundary conditions, which indicates the absence of massless particles.

If (as in Ising-like models) we assume that the intersection of a correlation line with a wall changes the sign of Γ , a small closed wall gives no contribution ($(-1)^2 = 1$) while a huge wall can give (-1) . If each link on the correlation line of total length L is assumed to intersect with a wall independently with probability p , we obtain

$$\langle \Gamma \rangle \sim \sum_{k=0}^L \binom{L}{k} (-1)^k p^k (1-p)^{L-k} = (1-2p)^L \sim e^{-mL}, \quad m = -\log(1-2p). \quad (97)$$

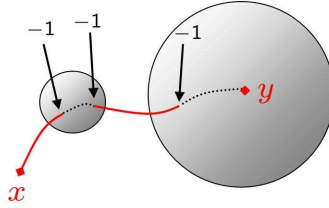


Figure 6: Disorder being caused by walls.

Our goal in this section is to elevate the above heuristic relation of disorder and a mass gap to a mathematically rigorous inequality.

Suppose that the explicit form of the correlation function Γ is given by

$$\Gamma_\mu(x; n) = \sum_{\alpha} f_{\alpha}(\phi_x) \overline{f_{\alpha}(\phi_{x+n\hat{\mu}})}, \quad n \in \mathbf{N}, \quad (98)$$

where f is an arbitrary function from the order-parameter space to \mathbf{C} endowed with generic indices $\{\alpha\}$. Here f_{α} is meant to specify, for example, the α -th component of a vector spin in $O(N)$ -like models, or the α -th matrix element of a matrix spin in PCM-like models. Note that $\langle \Gamma_{\mu}(x; n) \rangle$ is

independent of x due to the translational invariance of the system. An important requirement on Γ_μ is its invariance under G :

$$\sum_{\alpha} f_{\alpha}(\phi_x) \overline{f_{\alpha}(\phi_{x+n\hat{\mu}})} = \sum_{\alpha} f_{\alpha}(g\phi_x) \overline{f_{\alpha}(g\phi_{x+n\hat{\mu}})}, \quad g \in G. \quad (99)$$

Another requirement, which is *truly indispensable for all the following development*, is

$$\int_G dg f_{\alpha}(g\phi) = 0 \quad \text{for } \forall \phi. \quad (100)$$

If (100) were not satisfied, we should replace $f_{\alpha}(\phi)$ by $f'_{\alpha}(\phi) \equiv f_{\alpha}(\phi) - \int_G dg f_{\alpha}(g\phi)$.

Theorem 5. Assume (99), (100) and the existence of $k \in \mathbf{N}$ such that $L_{\mu} = 2^k n$, with L_{μ} the extent of Λ in the x^{μ} -direction.¹⁴ Then we have

$$\frac{|\langle \Gamma_{\mu}(x; n) \rangle|}{\langle \Gamma_{\mu}(x; 0) \rangle} \leq 2 \left\{ \frac{\langle \Gamma_{\mu}(x; 0)^2 \rangle}{\langle \Gamma_{\mu}(x; 0) \rangle^2} \right\}^{n/L_{\mu}} \left\{ 1 - \int_G dg \langle \mathcal{O}(g)[\mathcal{V}] \rangle \right\}^{n/L_{\mu}}, \quad (101)$$

where \mathcal{V} is a wall perpendicular to the x^{μ} -direction.¹⁵

Proof. We first decompose the correlation function into two parts:

$$\langle \Gamma_{\mu}(x; n) \rangle = \langle \Gamma_{\mu}(x; n)(1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \rangle + \langle \Gamma_{\mu}(x; n)\mathcal{F}_T(\mathbf{1})[\mathcal{V}] \rangle, \quad (102)$$

where T denotes the trivial representation, i.e. $\mathcal{F}_T(\mathbf{1})[\mathcal{V}] = \int_G dg \mathcal{O}(g)[\mathcal{V}]$. Our basic idea here is to apply the Schwarz inequality

$$|\langle F\theta G \rangle| \leq \langle F\theta F \rangle^{1/2} \langle G\theta G \rangle^{1/2} \quad (103)$$

to each term of (102). The procedure afterward is represented graphically in fig. 7 where $(1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}])$ is indicated by red segments and a blue line is drawn to guide the eye. Let $y \equiv x + n\hat{\mu}$ and $z \equiv x + 2n\hat{\mu}$. Consider a reflection θ about π (a hyperplane perpendicular to $\hat{\mu}$ and lying at y). Using (67), (80), (99) and (103) we get

$$\frac{|\langle \Gamma_{\mu}(x; n)(1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \rangle|}{\langle \Gamma_{\mu}(x; 0) \rangle} \quad (104)$$

$$= \frac{1}{\langle \Gamma_{\mu}(x; 0) \rangle} \left| \sum_{\alpha} \langle f_{\alpha}(\phi_x) \overline{f_{\alpha}(\phi_y)} (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \rangle \right| \quad (105)$$

$$\leq \frac{1}{\langle \Gamma_{\mu}(x; 0) \rangle} \sum_{\alpha} \left\langle f_{\alpha}(\phi_x) (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \cdot \theta \left[f_{\alpha}(\phi_x) (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \right] \right\rangle^{1/2} \left\langle \overline{f_{\alpha}(\phi_y)} \theta \left[\overline{f_{\alpha}(\phi_y)} \right] \right\rangle^{1/2} \quad (106)$$

$$= \frac{1}{\langle \Gamma_{\mu}(x; 0) \rangle} \sum_{\alpha} \left\langle f_{\alpha}(\phi_x) (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \overline{f_{\alpha}(\phi_z)} (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}^{\theta}]) \right\rangle^{1/2} \left\langle \overline{f_{\alpha}(\phi_y)} f_{\alpha}(\phi_y) \right\rangle^{1/2} \quad (107)$$

$$\leq \frac{1}{\langle \Gamma_{\mu}(x; 0) \rangle} \left\langle \sum_{\alpha} f_{\alpha}(\phi_x) \overline{f_{\alpha}(\phi_z)} (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \right\rangle^{1/2} \left\langle \sum_{\alpha} \overline{f_{\alpha}(\phi_y)} f_{\alpha}(\phi_y) \right\rangle^{1/2} \quad (108)$$

$$= \left\{ \frac{\langle \Gamma_{\mu}(x; 2n)(1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \rangle}{\langle \Gamma_{\mu}(x; 0) \rangle} \right\}^{1/2}. \quad (109)$$

¹⁴ It does not seem to be very restrictive; in general we believe in the existence of the limit $L_{\mu} \rightarrow \infty$ for physical observables independent of the way we take $L_{\mu} \rightarrow \infty$.

¹⁵ $0 < \langle \Gamma_{\mu}(n) \rangle$ can be proved by the site-reflection positivity if n is even.

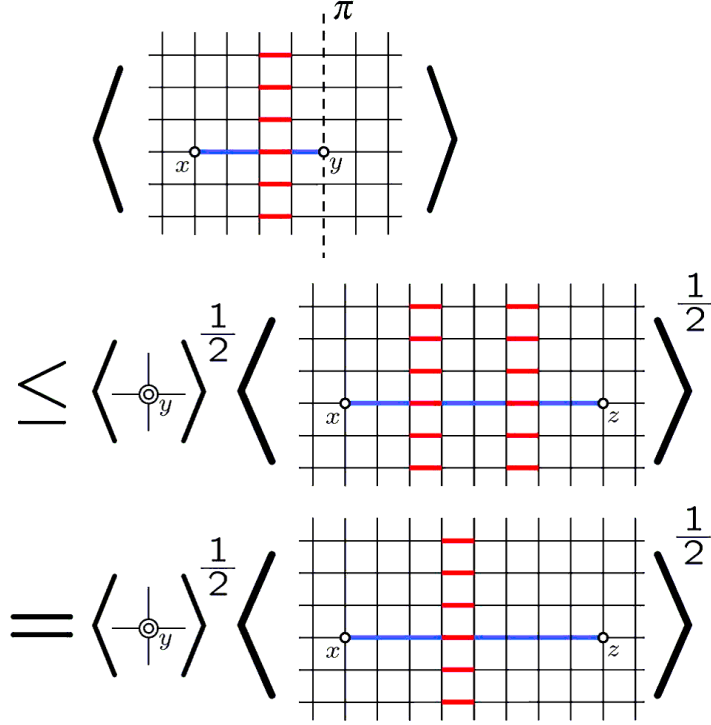


Figure 7: Schwarz inequality enables us to double the distance of points in the correlation function.

Iterating this procedure for $(k - 1)$ times yields (note $L_\mu = 2^k n$)

$$\frac{|\langle \Gamma_\mu(n)(1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \rangle|}{\langle \Gamma_\mu(0) \rangle} \leq \left\{ \frac{\langle \Gamma_\mu(L_\mu/2)(1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \rangle}{\langle \Gamma_\mu(0) \rangle} \right\}^{2n/L_\mu}. \quad (110)$$

Finally, let us define the reflection θ w.r.t. the hyperplane π which runs through x and $x + (L_\mu/2)\hat{\mu}$ (thus $\theta[x] = x$ and $\theta[x + (L_\mu/2)\hat{\mu}] = x + (L_\mu/2)\hat{\mu}$, see fig.8). Then we obtain

$$\frac{\langle \Gamma_\mu(x; L_\mu/2)(1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \rangle}{\langle \Gamma_\mu(x; 0) \rangle} \quad (111)$$

$$\leq \frac{\langle \Gamma_\mu(x; L_\mu/2)\theta[\Gamma_\mu(x; L_\mu/2)] \rangle^{1/2} \langle (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}^\theta]) \rangle^{1/2}}{\langle \Gamma_\mu(x; 0) \rangle} \quad (112)$$

$$\leq \frac{\langle \Gamma_\mu(x; 0)^2 \rangle^{1/2} \{1 - \langle \mathcal{F}_T(\mathbf{1})[\mathcal{V}] \rangle\}^{1/2}}{\langle \Gamma_\mu(x; 0) \rangle} \left(= \left\{ \frac{\langle \Gamma_\mu(x; 0)^2 \rangle}{\langle \Gamma_\mu(x; 0) \rangle^2} \right\}^{1/2} \{1 - \langle \mathcal{F}_T(\mathbf{1})[\mathcal{V}] \rangle\}^{1/2} \right), \quad (113)$$

thus

$$0 \leq \frac{\langle \Gamma_\mu(x; n)(1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}]) \rangle}{\langle \Gamma_\mu(x; 0) \rangle} \leq \left\{ \frac{\langle \Gamma_\mu(x; 0)^2 \rangle}{\langle \Gamma_\mu(x; 0) \rangle^2} \right\}^{n/L_\mu} \{1 - \langle \mathcal{F}_T(\mathbf{1})[\mathcal{V}] \rangle\}^{n/L_\mu}. \quad (114)$$

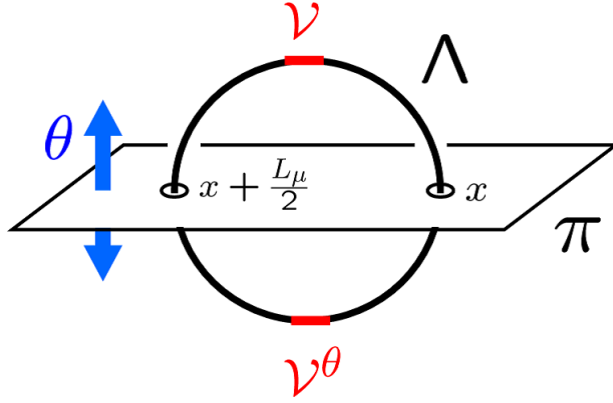


Figure 8: θ is a reflection about π that bisects Λ . Here Λ is shown as a one-dimensional chain for simplicity.

In deriving (113) we used $\langle \Gamma_\mu(x; L_\mu/2) \theta[\Gamma_\mu(x; L_\mu/2)] \rangle \leq \langle \Gamma_\mu(x; 0)^2 \rangle$. This can be shown as follows:

$$\langle \Gamma_\mu(x; L_\mu/2) \theta[\Gamma_\mu(x; L_\mu/2)] \rangle \quad (115)$$

$$= \sum_{\alpha, \beta} \left\langle f_\alpha(\phi_x) \overline{f_\alpha(\phi_{x+(L_\mu/2)\hat{\mu}})} \cdot \theta \left[f_\beta(\phi_x) \overline{f_\beta(\phi_{x+(L_\mu/2)\hat{\mu}})} \right] \right\rangle \quad (116)$$

$$= \sum_{\alpha, \beta} \left\langle f_\alpha(\phi_x) \overline{f_\alpha(\phi_{x+(L_\mu/2)\hat{\mu}})} \overline{f_\beta(\phi_x)} f_\beta(\phi_{x+(L_\mu/2)\hat{\mu}}) \right\rangle \quad (117)$$

$$\leq \sum_{\alpha, \beta} \left\langle f_\alpha(\phi_x) \overline{f_\beta(\phi_x)} \cdot \theta \left[f_\alpha(\phi_x) \overline{f_\beta(\phi_x)} \right] \right\rangle^{1/2} \left\langle \overline{f_\alpha(\phi_{x+(L_\mu/2)\hat{\mu}})} f_\beta(\phi_{x+(L_\mu/2)\hat{\mu}}) \cdot \theta \left[\overline{f_\alpha(\phi_{x+(L_\mu/2)\hat{\mu}})} f_\beta(\phi_{x+(L_\mu/2)\hat{\mu}}) \right] \right\rangle^{1/2} \quad (118)$$

$$= \sum_{\alpha, \beta} \langle |f_\alpha(\phi_x)|^2 |f_\beta(\phi_x)|^2 \rangle^{1/2} \langle |f_\alpha(\phi_{x+(L_\mu/2)\hat{\mu}})|^2 |f_\beta(\phi_{x+(L_\mu/2)\hat{\mu}})|^2 \rangle^{1/2} \quad (119)$$

$$\leq \left\langle \sum_{\alpha, \beta} |f_\alpha(\phi_x)|^2 |f_\beta(\phi_x)|^2 \right\rangle^{1/2} \left\langle \sum_{\alpha, \beta} |f_\alpha(\phi_{x+(L_\mu/2)\hat{\mu}})|^2 |f_\beta(\phi_{x+(L_\mu/2)\hat{\mu}})|^2 \right\rangle^{1/2} \quad (120)$$

$$= \langle \Gamma_\mu(x; 0)^2 \rangle^{1/2} \langle \Gamma_\mu(x + (L_\mu/2); 0)^2 \rangle^{1/2} = \langle \Gamma_\mu(x; 0)^2 \rangle. \quad (121)$$

Next we have to estimate $\frac{\langle \Gamma_\mu(x; n) \mathcal{F}_T(\mathbf{1})[\mathcal{V}] \rangle}{\langle \Gamma_\mu(x; 0) \rangle}$. The outline is similar to the previous case, but additional intricacies occur.

$$0 \leq \langle \Gamma_\mu(x; n) \mathcal{F}_T(\mathbf{1})[\mathcal{V}] \rangle \quad (122)$$

$$= \sum_{\alpha} \int_G dg \langle f_\alpha(\phi_x) \overline{f_\alpha(\phi_y)} \mathcal{O}(g) [\mathcal{V}] \rangle \quad (123)$$

Let us consider another wall \mathcal{V}' that is parallel to \mathcal{V} and bookends x with \mathcal{V} (see fig.9). Moving \mathcal{V} to \mathcal{V}' passing over ϕ_x and using (100), we get

$$= \sum_{\alpha} \int_G dg \langle f_\alpha(g\phi_x) \overline{f_\alpha(\phi_y)} \mathcal{O}(g) [\mathcal{V}'] \rangle \quad (124)$$

$$= \sum_{\alpha} \int_G dg \left\langle f_\alpha(g\phi_x) \overline{f_\alpha(\phi_y)} \left(\mathcal{O}(g) [\mathcal{V}'] - \mathcal{F}_T(\mathbf{1}) [\mathcal{V}'] \right) \right\rangle. \quad (125)$$

This step (from (124) to (125)) is the most nontrivial operation in this proof. Using (103) w.r.t. the

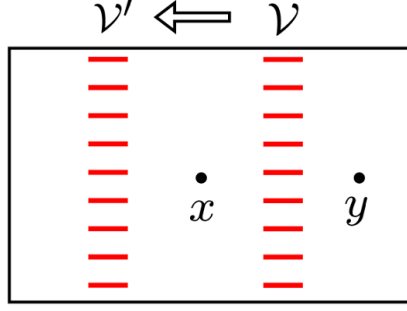


Figure 9: By redefinition of variables, \mathcal{V} is moved to \mathcal{V}' , passing over the site x .

hyperplane which runs through y , with $z = \theta[x]$, yields

$$= \sum_{\alpha} \int_G dg \left\langle f_{\alpha}(g\phi_x) \left(\mathcal{O}(g)[\mathcal{V}'] - \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \right) \cdot \theta \left[f_{\alpha}(g\phi_x) \left(\mathcal{O}(g)[\mathcal{V}'] - \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \right) \right] \right\rangle^{1/2} \langle \overline{f_{\alpha}(\phi_y)} f_{\alpha}(\phi_y) \rangle^{1/2} \quad (126)$$

$$= \sum_{\alpha} \int_G dg \left\langle f_{\alpha}(g\phi_x) \left(\mathcal{O}(g)[\mathcal{V}'] - \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \right) \overline{f_{\alpha}(g\phi_z)} \left(\mathcal{O}(g^{-1})[\mathcal{V}'^{\theta}] - \mathcal{F}_T(\mathbf{1})[\mathcal{V}'^{\theta}] \right) \right\rangle^{1/2} \langle \overline{f_{\alpha}(\phi_y)} f_{\alpha}(\phi_y) \rangle^{1/2} \quad (127)$$

$$= \sum_{\alpha} \int_G dg \left\langle f_{\alpha}(g\phi_x) \overline{f_{\alpha}(g\phi_z)} \left(1 - \mathcal{O}(g)[\mathcal{V}'] \mathcal{F}_T(\mathbf{1})[\mathcal{V}'^{\theta}] - \mathcal{O}(g^{-1})[\mathcal{V}'^{\theta}] \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] + \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \right) \right\rangle^{1/2} \times \langle \overline{f_{\alpha}(\phi_y)} f_{\alpha}(\phi_y) \rangle^{1/2} \quad (128)$$

$$\leq \langle \Gamma_{\mu}(y; 0) \rangle^{1/2} \int_G dg \left\langle \Gamma_{\mu}(x; 2n) \left(1 - \mathcal{O}(g)[\mathcal{V}'] \mathcal{F}_T(\mathbf{1})[\mathcal{V}'^{\theta}] - \mathcal{O}(g^{-1})[\mathcal{V}'^{\theta}] \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] + \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \right) \right\rangle^{1/2} \quad (129)$$

$$\leq \langle \Gamma_{\mu}(x; 0) \rangle^{1/2} \left[\int_G dg \left\langle \Gamma_{\mu}(x; 2n) \left(1 - \mathcal{O}(g)[\mathcal{V}'] \mathcal{F}_T(\mathbf{1})[\mathcal{V}'^{\theta}] - \mathcal{O}(g^{-1})[\mathcal{V}'^{\theta}] \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] + \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \right) \right\rangle \right]^{1/2} \quad (130)$$

$$= \langle \Gamma_{\mu}(x; 0) \rangle^{1/2} \left[\langle \Gamma_{\mu}(x; 2n) \rangle - \langle \Gamma_{\mu}(x; 2n) \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \mathcal{F}_T(\mathbf{1})[\mathcal{V}'^{\theta}] \rangle - \langle \Gamma_{\mu}(x; 2n) \mathcal{F}_T(\mathbf{1})[\mathcal{V}'^{\theta}] \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \rangle + \langle \Gamma_{\mu}(x; 2n) \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \rangle \right]^{1/2} \quad (131)$$

$$= \langle \Gamma_{\mu}(x; 0) \rangle^{1/2} \langle \Gamma_{\mu}(x; 2n) (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}']) \rangle^{1/2}, \quad (132)$$

$$\therefore \frac{\langle \Gamma_{\mu}(x; n) \mathcal{F}_T(\mathbf{1})[\mathcal{V}] \rangle}{\langle \Gamma_{\mu}(x; 0) \rangle} \leq \left\{ \frac{\langle \Gamma_{\mu}(x; 2n) (1 - \mathcal{F}_T(\mathbf{1})[\mathcal{V}']) \rangle}{\langle \Gamma_{\mu}(x; 0) \rangle} \right\}^{1/2}. \quad (133)$$

Combining this result with (114) yields

$$\frac{\langle \Gamma_{\mu}(x; n) \mathcal{F}_T(\mathbf{1})[\mathcal{V}] \rangle}{\langle \Gamma_{\mu}(x; 0) \rangle} \leq \left\{ \frac{\langle \Gamma_{\mu}(x; 0)^2 \rangle}{\langle \Gamma_{\mu}(x; 0) \rangle^2} \right\}^{n/L_{\mu}} \{1 - \langle \mathcal{F}_T(\mathbf{1})[\mathcal{V}'] \rangle\}^{n/L_{\mu}}. \quad (134)$$

(114) and (134) lead to (101). \square

Here are a few comments:

- $\langle \mathcal{O}(g) \rangle$ is a class function on G . It implies that when G is a compact connected Lie group we can rewrite the integral in the r.h.s. of (101) as an integral over the maximal torus of G with a proper Jacobian, using the Weyl's integration formula [33].
- According to (101), *the exponential decay of the wall free energy, i.e.*

$$\langle \mathcal{O}(g)[\mathcal{V}] \rangle = 1 - O(e^{-\rho(g)L_\mu}) \quad \text{with } \rho(g) > 0, \quad \forall g \in G, \quad (135)$$

is a sufficient condition for the existence of a mass gap. Indeed if we define $\bar{\rho}[G] \equiv \min_{g \in G} \rho(g)$ and take the limit $L_\mu \rightarrow \infty$, the r.h.s. of (101) converges to $O(1) \times e^{-\bar{\rho}[G]n}$, thus $\bar{\rho}[G]$ gives a lower bound for the mass gap. It is also obvious that $\bar{\rho}[G] \leq \bar{\rho}[H]$ follows if $H \subset G$. $\langle \mathcal{O}(g)[\mathcal{V}] \rangle$ would be calculable by, for example, Monte Carlo Simulations and weak-coupling-, strong-coupling-, $1/d$ - and $1/N$ -expansions.

- We proved the theorem on a square lattice, but it can be easily generalized to other lattices such as a triangular lattice.
- Note that theorem 5 is derived with no knowledge of the interaction except for the reflection positivity. The strength of interaction can be made anisotropic, since it respects reflection positivity. And it is also correct in arbitrary dimensions.

3.3 Examples

There are many classes of lattice systems to which our theorem is applicable. One is the class of *coset models*, where the field takes values in G/H with G an arbitrary Lie group and H its closed subgroup. We present three explicit examples.

Example 1. *The $O(N)$ Heisenberg model.*

The partition function is given by

$$Z = \int \prod_{y \in \Lambda} d\vec{\phi}_y \exp \left(\beta \sum_{x, \mu} \vec{\phi}_x \cdot \vec{\phi}_{x+\hat{\mu}} \right), \quad \vec{\phi} \in S^{N-1} = O(N)/O(N-1). \quad (136)$$

The most natural definition of a correlation function is

$$\Gamma_\mu(x; n) = \vec{\phi}_x \cdot \vec{\phi}_{x+n\hat{\mu}}. \quad (137)$$

In this case Z_2 is obviously the smallest possible invariance group; of course, $O(N)$ itself can also be chosen. One can check that the requirements (99), (100) are met for both of them. If one is going to choose other arbitrary subgroup of $O(N)$, the correlation function should be redefined properly as explained below (100). Since the length of spin is normalized, the prefactor in (101) becomes 1.

Example 2. *The CP^{N-1} model.*

The partition function is given by¹⁶

$$Z = \int \prod_{y \in \Lambda} dP_y \exp \left(\beta \sum_{x, \mu} \text{Tr} [P_x P_{x+\hat{\mu}}] \right), \quad P \in CP^{N-1} = U(N)/(U(1) \times U(N-1)). \quad (138)$$

P is an $N \times N$ matrix and obeys $P_x^2 = P_x$, $P_x^\dagger = P_x$, $\text{Tr} P_x = 1$. (An alternative way is to express $P_{ij} = z_i^* z_j$, where $\vec{z} \in \mathbb{C}^N$ and $|\vec{z}|^2 = 1$.) The action is invariant under the global transformation $P_x \rightarrow$

¹⁶See also ref.[34].

$\Omega P_x \Omega^\dagger$ with $\Omega \in SU(N)$.¹⁷ In two dimension, the model is believed to possess a nonperturbatively generated mass gap for which, however, no rigorous result is available. Defining an appropriate correlation function needs some care; we define

$$\Gamma_\mu(x; n) = \text{Tr} \left\{ \left[P_x - \frac{\mathbf{1}_N}{N} \right] \left[P_{x+n\hat{\mu}} - \frac{\mathbf{1}_N}{N} \right] \right\} = \text{Tr} [P_x P_{x+n\hat{\mu}}] - \frac{1}{N}, \quad (139)$$

where $\mathbf{1}_N$ denotes the unit matrix of size $N \times N$. Then it is easy to confirm (100) for $G = SU(N)$:

$$\int_{SU(N)} d\Omega \left[\Omega P \Omega^\dagger - \frac{\mathbf{1}_N}{N} \right]_{ij} = P_{kl} \int_{SU(N)} d\Omega \Omega_{ik} \Omega_{lj}^\dagger - \frac{\delta_{ij}}{N} \quad (140)$$

$$= P_{kl} \frac{\delta_{ij} \delta_{kl}}{N} - \frac{\delta_{ij}}{N} = 0. \quad (141)$$

Example 3. $G \times G$ principal chiral model (PCM).

Here G is an arbitrary compact group. The partition function is given by

$$Z = \int \prod_{y \in \Lambda} dU_y \exp \left(\beta \sum_{x, \mu \subset \Lambda} \text{Re Tr} [U_x (U_{x+\hat{\mu}})^{-1}] \right), \quad U_x \in G. \quad (142)$$

This model is invariant under a global $G \times G$ transformation $U \rightarrow g_L U g_R^{-1}$. The corresponding twisted partition function is given, in agreement with the original definition (62), by (\mathcal{V} : wall)

$$Z(g) = \int \prod_{y \in \Lambda} dU_y \exp \left(\beta \sum_{x, \mu \subset \Lambda \setminus \mathcal{V}} \text{Re Tr} [U_x (U_{x+\hat{\mu}})^{-1}] + \beta \sum_{x, \mu \subset \mathcal{V}} \text{Re Tr} [U_x (g U_{x+\hat{\mu}})^{-1}] \right), \quad g \in G. \quad (143)$$

The correlation function is defined as

$$\Gamma_\mu(x; n) = \chi_R(U_x U_{x+n\hat{\mu}}^{-1}), \quad (144)$$

where R denotes an irreducible unitary representation of G . If we choose G itself, then (100) is satisfied if and only if R is a nontrivial representation. This is a direct consequence of the Schur orthogonality relation in representation theory; see Corollary 4.10 in ref.[33]. If $G = SU(N)$ and we choose Z_N , then (100) is satisfied iff R has a nonzero N -ality. Anyway the prefactor in (101) becomes 1. We emphasize that G need not have a nontrivial center.

As an example other than coset models, we only present

Example 4. $SU(N_f) \times SU(N_f)$ linear sigma model.

Usually this model is used to study the chiral phase transition [35]. On the lattice, the partition function is given by

$$Z = \int \prod_{y \in \Lambda} d\Phi_y \exp \left(\beta_1 \sum_{x, \mu} \text{Re Tr} (\Phi_x \Phi_{x+\hat{\mu}}^\dagger) - \beta_2 \sum_{x \in \Lambda} \text{Tr} (\Phi_x^\dagger \Phi_x) - \lambda_1 \sum_{x \in \Lambda} [\text{Tr} (\Phi_x^\dagger \Phi_x)]^2 - \lambda_2 \sum_{x \in \Lambda} \text{Tr} (\Phi_x^\dagger \Phi_x)^2 \right), \quad (145)$$

$$\Phi_x \in M(N_f, \mathbb{C}). \quad (146)$$

This model is invariant under $\Phi \rightarrow e^{i\alpha} g_L \Phi g_R^\dagger$ with $e^{i\alpha} \in U_A(1)$ and $g_L, g_R \in SU(N_f)$.

¹⁷The *true* symmetry is $SU(N)/Z_N$, since $\Omega \in Z_N$ does not change P_x at all.

It might be the case that *whether the upper bound (101) gives an exponential decay or not depends on the choice of G* , a point worth further study.

* * * *

Although we proved theorem 5 only in the case of the nearest-neighbor interaction, **we can prove it even in the presence of non-nearest-neighbor and multi-site interactions if some appropriate conditions are satisfied.** To make the argument concrete, let us consider the $SU(N) \times SU(N)$ PCM and suppose that $Z_N \subset SU(N)$ was chosen as a symmetry group for theorem 5. There are two crucial conditions: one is site-reflection positivity and the other is that twists and their algebra (see 3.1) should remain well-defined. Here is a partial list of possible extensions (on a square lattice):

1. The multi-site interaction term between four variables on the same plaquette,

$$\text{Re } \chi_R(U_x U_{x+\hat{\mu}}^\dagger U_{x+\hat{\mu}+\hat{\nu}} U_{x+\hat{\nu}}^\dagger), \quad (147)$$

can be added to the action without spoiling theorem 5 if the N -ality of R is 0.

2. The non-nearest-neighbor interaction term $\text{Re } \chi_R(U_x U_{x+2\hat{\mu}}^\dagger)$ can be added to the action if the N -ality of R is 0 and the coefficient in front of it is positive. (If the distance is larger than two lattice spacings, or if the coefficient is negative, then the site-reflection positivity becomes hard to prove.)
3. The non-nearest-neighbor interaction term $\text{Re } \chi_R(U_x U_{x+\hat{\mu}+\hat{\nu}}^\dagger)$, $\mu \neq \nu$, can be added to the action if the N -ality of R is 0.

* * * *

Note that our proved inequality may fail to be useful in phases other than the disordered phase, even though it is correct in any phases. Consider a spin system with a global symmetry group G in a d -dimensional box of size $L_1 \times \cdots \times L_d$ whose boundary condition is twisted by $g \in G$ in the x^1 -direction and otherwise periodic. Let Z^g denote the twisted partition function and set $L_\perp \equiv \prod_{k=2}^d L_k$. Based on our experience in gauge theories, we generally expect following behaviors of Z^g/Z in various phases:

$$\begin{aligned} Z^g/Z &\approx \exp(-x L_\perp \exp(-y L_1)) && \text{(Disordered phase)} \\ Z^g/Z &\approx \exp(-z L_\perp) && \text{(Ordered phase)} \\ Z^g/Z &\approx \exp(-w L_\perp / L_1) && \text{(Massless phase)} \end{aligned} \quad (148)$$

where x, y, z and w are functions of the coupling constants and the choice of g . Letting α denote the value of $(1 - Z^g/Z)^{1/L_1}$ in the thermodynamic limit, we have $\alpha = e^{-y}$ in the disordered phase, while $\alpha = 1$ in the other cases so that the r.h.s. of (101) tends to a constant independent of n . So the bottom line is that algebraic decay of correlation function cannot be inferred from the behavior of the r.h.s. in general.

Finally we comment on the formal difference between the inequality derived by Kovács and Tomboulis in refs.[19, 20] and ours in the two-dimensional $SU(2) \times SU(2)$ PCM. Their result is

$$\langle \Gamma_\mu(x; n) \rangle \Big|_{L_\mu=\infty} \leq \frac{Z_n(+, +) - Z_n(-, -)}{Z_n(+, +) + Z_n(+, -) + Z_n(-, +) + Z_n(-, -)}, \quad (149)$$

where $Z_n(\tau_1, \tau_2)$ ($\tau_{1,2} = \pm 1$) is the partition function on the lattice of size $n \times n$ with a twist τ_μ operated in the x^μ -direction.

On the other hand, our result ((101) with $G = Z_2$) gives

$$\langle \Gamma_\mu(x; n) \rangle \leq 2 \{1 - \langle \mathcal{F}_0[\mathcal{V}] \rangle\}^{n/L_\mu} \quad (150)$$

$$= 2 \left\{ \frac{1}{2} \left[\frac{Z_{L_\mu}(+, +) - Z_{L_\mu}(+, -)}{Z_{L_\mu}(+, +)} \right] \right\}^{n/L_\mu}. \quad (151)$$

Note that in the latter, both r.h.s. and l.h.s. are estimated on the lattice of size L_μ . In both formulas the correlation function is assumed to be in the fundamental representation. Although they look different, both relates the exponential suppression of the wall free energy (in the thermodynamic limit) to the mass gap, thus their physical contents are totally consistent.

3.4 Demonstration in 1D PCM and 2D square Ising model

Let us verify the proved inequality (101) explicitly in the $G \times G$ PCM in one-dimension, with G an arbitrary compact group. The partition function of the model on a periodic chain of length L is given by

$$Z_\Lambda \equiv \int \prod_{k=1}^L dU_k \exp \left(\beta \sum_{i=1}^L \text{Re Tr} (U_i U_{i+1}^{-1}) \right), \quad \beta > 0, U_i \in G, \quad (152)$$

$$= \int \prod_{k=1}^L dU_k \prod_{i=1}^L \left[\sum_r d_r F_r \chi_r(U_i U_{i+1}^{-1}) \right], \quad (153)$$

where \sum_r runs over all irreducible unitary representations of G . $F_r = F_{\bar{r}} > 0$ follows from the reflection positivity and reality of the action. Let us define $c_r \equiv F_r/F_0$ for later convenience (0 denotes the trivial representation). Straightforward calculation yields

$$Z_\Lambda = \sum_r d_r^2 (F_r)^L. \quad (154)$$

The twisted partition function Z_Λ^g is similarly given by

$$Z_\Lambda^g \equiv \int \prod_{k=1}^L dU_k \left[\sum_{r'} d_{r'} F_{r'} \chi_{r'}(g U_1 U_2^{-1}) \right] \prod_{i=2}^L \left[\sum_r d_r F_r \chi_r(U_i U_{i+1}^{-1}) \right], \quad g \in G', \quad (155)$$

$$= \sum_r d_r \chi_r(g) (F_r)^L, \quad (156)$$

where $G' \subset G$ is an arbitrary subgroup of G . (Z_Λ^g reduces to Z_Λ for $g = \mathbf{1}$, as it should be.) Hence we get

$$\lim_{L \rightarrow \infty} \left\{ 1 - \int_{G'} dg \langle \mathcal{O}(g) \rangle \right\}^{n/L} = \lim_{L \rightarrow \infty} \left\{ \frac{\sum_r' d_r^2 (c_r)^L}{\sum_r d_r^2 (c_r)^L} \right\}^{n/L} \quad (157)$$

$$= (c_{r'})^n. \quad (158)$$

Here \sum_r' is defined as a sum over all representations of G which are nontrivial w.r.t. G' , and $c_{r'}$ is defined as the largest one among $\{c_r \mid r \text{ is nontrivial w.r.t. } G'\}$.

Next we define the correlation function as $\Gamma(n) = \chi_R(U_0 U_n^{-1})$. Then (100) requires R to be nontrivial w.r.t. G . After straightforward calculation we get

$$\frac{\langle \Gamma(n) \rangle}{\langle \Gamma(0) \rangle} = \frac{1}{d_R} \langle \chi_R(U_0 U_n^{-1}) \rangle = (c_R)^n \quad (159)$$

in the thermodynamic limit ($L \rightarrow \infty$). Since $c_R \leq c_{r'}$ is obvious from their definitions, we conclude from (158) and (159) that the inequality (101) certainly holds at least in the limit $L \rightarrow \infty$.¹⁸

* * * *

As a next example let us take the two-dimensional Ising model on a square lattice. The partition function of the model is given by

$$Z_\Lambda = \int \prod_{k \in \Lambda} d\sigma_k \exp \left(\sum_{\mu=1}^{L_1} \sum_{\nu=1}^{L_2} (a \sigma_{\mu\nu} \sigma_{\mu+1,\nu} + b \sigma_{\mu\nu} \sigma_{\mu,\nu+1}) \right), \quad (160)$$

where $\sigma_{\mu\nu}$ is the Ising spin located at the site (μ, ν) and $\int d\sigma \equiv \frac{1}{2} \sum_{\sigma=\pm 1}$. Periodic boundary conditions are imposed so that $\sigma_{1,\nu} = \sigma_{L_1+1,\nu}$, $\sigma_{\mu,1} = \sigma_{\mu,L_2+1}$. We assume $a > 0$, $b > 0$. Let us focus on the high temperature (disorder) phase of the model.

The exact asymptotic form of the two-point correlation function is known [36] and the mass gap (or inverse correlation length) $M \equiv 2(\bar{a} - b)$, where \bar{a} is the *dual temperature* defined by

$$\sinh 2a \sinh 2\bar{a} = 1. \quad (161)$$

\bar{b} is defined in the same way.

To estimate the free energy of walls we need explicit formulae for twisted and untwisted partition functions. Here we use the expressions due to Kastening [37], which in our notation read

$$Z_\Lambda = \frac{1}{2} [2 \sinh(2a)]^{L_1 L_2 / 2} \times \left\{ \prod_{k=1}^{L_2} \left[2 \cosh \left(\frac{L_1}{2} \gamma_{2k-1} \right) \right] + \prod_{k=1}^{L_2} \left[2 \sinh \left(\frac{L_1}{2} \gamma_{2k-1} \right) \right] + \prod_{k=1}^{L_2} \left[2 \cosh \left(\frac{L_1}{2} \gamma_{2k-2} \right) \right] - \prod_{k=1}^{L_2} \left[2 \sinh \left(\frac{L_1}{2} \gamma_{2k-2} \right) \right] \right\}, \quad (162)$$

$$Z_\Lambda^{(-)} = \frac{1}{2} [2 \sinh(2a)]^{L_1 L_2 / 2} \times \left\{ \prod_{k=1}^{L_2} \left[2 \cosh \left(\frac{L_1}{2} \gamma_{2k-1} \right) \right] + \prod_{k=1}^{L_2} \left[2 \sinh \left(\frac{L_1}{2} \gamma_{2k-1} \right) \right] - \prod_{k=1}^{L_2} \left[2 \cosh \left(\frac{L_1}{2} \gamma_{2k-2} \right) \right] + \prod_{k=1}^{L_2} \left[2 \sinh \left(\frac{L_1}{2} \gamma_{2k-2} \right) \right] \right\}. \quad (163)$$

$Z_\Lambda^{(-)}$ is the twisted partition function; more precisely, it is a partition function on a lattice which is antiperiodic in x^1 -direction and periodic in x^2 -direction. (Note that this boundary condition is equivalent to the existence of a wall wrapping around a periodic lattice in x^2 -direction.) $\gamma_k > 0$ is defined by

$$\cosh \gamma_k = \cosh 2\bar{a} \cosh 2b - \cos \frac{\pi k}{L_2} \sinh 2\bar{a} \sinh 2b. \quad (164)$$

¹⁸(101) should hold for finite L too, but expressions of both sides of (101) become highly complicated for finite L and verification seems to be hard.

The inequality to be checked, namely (101) for the square Ising model, is given by

$$\langle \sigma_0 \sigma_n \rangle_\Lambda \leq 2 \left\{ \frac{1}{2} \left(1 - \frac{Z_\Lambda^{(-)}}{Z_\Lambda} \right) \right\}^{n/L_1}. \quad (165)$$

$\langle \dots \rangle_\Lambda$ denotes the expectation value measured on a finite lattice ($= \Lambda$). Similarly $\langle \dots \rangle_\infty$ denotes an expectation value in the thermodynamic limit. For simplicity we calculate not $\frac{Z_\Lambda^{(-)}}{Z_\Lambda}$ but

$$\frac{Z_\Lambda - Z_\Lambda^{(-)}}{Z_\Lambda + Z_\Lambda^{(-)}} = \frac{1 - \prod_{k=1}^{L_2} \tanh \left(\frac{L_1}{2} \gamma_{2k-2} \right)}{\prod_{k=1}^{L_2} \frac{\cosh \left(\frac{L_1}{2} \gamma_{2k-1} \right)}{\cosh \left(\frac{L_1}{2} \gamma_{2k-2} \right)} + \prod_{k=1}^{L_2} \frac{\sinh \left(\frac{L_1}{2} \gamma_{2k-1} \right)}{\cosh \left(\frac{L_1}{2} \gamma_{2k-2} \right)}}. \quad (166)$$

Considering that¹⁹ $0 < \gamma_0 = 2(\bar{a} - b)$ is the smallest among $\{\gamma_k\}$, we obtain, after some algebra,

$$\lim_{L_1 \rightarrow \infty} \left(1 - \frac{Z_\Lambda^{(-)}}{Z_\Lambda} \right)^{1/L_1} = \exp \left\{ - \left(\gamma_0 + \frac{1}{2} \sum_{k=0}^{2L_2-1} (-1)^{k+1} \gamma_k \right) \right\}. \quad (167)$$

Since $\sum_{k=0}^{2L_2-1} (-1)^{k+1} \gamma_k = O\left(\frac{1}{L_2}\right)$ for $L_2 \gg 1$, we get

$$\lim_{L_2 \rightarrow \infty} \lim_{L_1 \rightarrow \infty} \left[\text{r.h.s. of (165)} \right] = 2e^{-\gamma_0 n}. \quad (168)$$

We compare this result with the asymptotic form of the exact two point function in the high temperature phase [36]:

$$\langle \sigma_0 \sigma_n \rangle_\infty = f(a, b) \frac{1}{\sqrt{n}} e^{-\gamma_0 n} \times \left[1 + O\left(\frac{1}{n}\right) \right] \quad \text{for } n \gg 1, \quad (169)$$

where the factor $f(a, b)$ is independent of n .

(168) and (169) tells that the exponential decay rates of both sides *coincide exactly* for all values of (a, b) when the system is in the disorder phase. This result suggests that our inequality might be a rather accurate one in general.

3.5 Demonstration in 2D triangular Ising model

Our next example is the two-dimensional Ising model on a *triangular* lattice. In the square Ising model, exact expression for mass gap was already known, thus the value of our theorem is obscured. However, as for the triangular Ising model, an exact expression for mass gap is not known except for special cases, so (unlike in the previous section) *the results we give in this section are essentially new*.

We consider a lattice of size $L_1 \times L_2$ with periodic boundary conditions.²⁰ See fig.10 for an example with $L_1 = 6$ and $L_2 = 3$; upper and lower edges painted in blue should be identified, and also right and left edges painted in red should be identified. (This lattice is the one that appeared in the seminal work of Houtappel [38] in which an analytic formula for the triangular Ising model was obtained for

¹⁹ $0 < \gamma_0$ stems from the fact that now the system is in the high temperature (disorder) phase.

²⁰ L_1 denotes the number of triangles. It is not the *actual length* of the lattice.

the first time.) Another triangular lattice commonly used in the literature is depicted in fig.11, where edges are again colored for the purpose of indicating the periodic structure of the lattice. It is easy to prove that these lattices are equivalent *if and only if* L_1 is a multiple of $2L_2$. See fig.12 for illustration of this fact. Numbers are written to guide the eye; edges assigned with the same number should be identified. Hereafter we will assume this condition, but this is only a technical assumption and not essential for we will be interested in the limit $L_1 \rightarrow \infty$. The reason we did not start with the lattice in fig.11 is because it does not allow for simple use of reflection positivity.

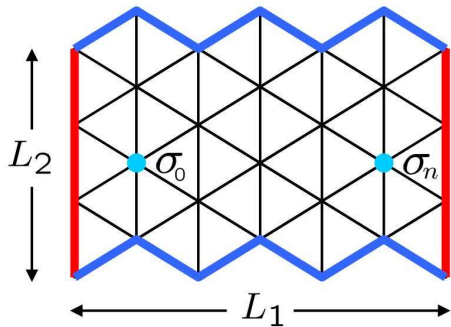


Figure 10: A triangular lattice of size $L_1 \times L_2$ with periodic boundary conditions; the red and the blue ends are identified, respectively. This lattice is symmetric w.r.t. each of vertical axes.

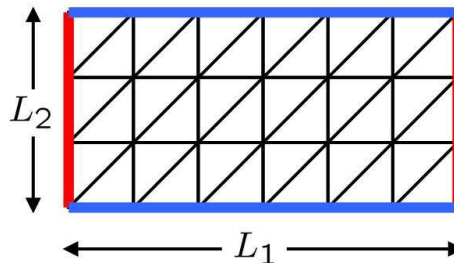


Figure 11: A triangular lattice constructed from a square lattice by addition of diagonal edges. Its periodic structure is indicated by coloring as in fig.10.

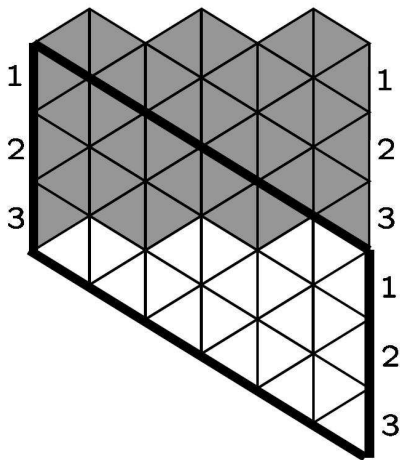


Figure 12: An illustration of the fact that those lattices given in figs.10 and 11 are equivalent iff L_1 is a multiple of $2L_2$. Here $L_1 = 6$ and $L_2 = 3$.

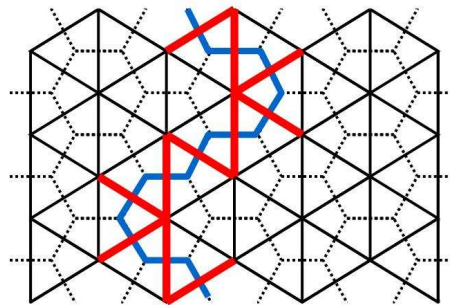


Figure 13: A Z_2 -twist represented by a blue loop on the dual (hexagonal) lattice. On the original (triangular) lattice, it is represented by a stacked set of links (colored in red) with couplings of opposite sign.

Let us define a twist on a planar triangular lattice. A twist is a closed loop on the dual lattice, and the dual of a triangular lattice is a honeycomb (or hexagonal) lattice as shown in fig.13. Note that the blue line in fig.13 is a closed loop owing to the periodic structure of the lattice. It becomes, on the original lattice, a stacked set of links with a coupling constant of opposite sign, which is depicted as a set of red links in fig.13. Note that introducing a twist to a periodic lattice as in fig.13 is equivalent to imposing an anti-periodic boundary condition in the horizontal direction.

Let us remember that it is not the number of the walls but rather the number *mod 2* of them that is physically relevant. This ‘ Z_2 conservation’ of walls is a direct consequence of $\sigma^2 = 1$ in the present model, and we can show it explicitly by a sequence of changes of variables $\sigma \rightarrow -\sigma$. For instance, the partition function containing one wall and that containing three walls agree completely as illustrated in fig.14 in which red segments represent twisted links.

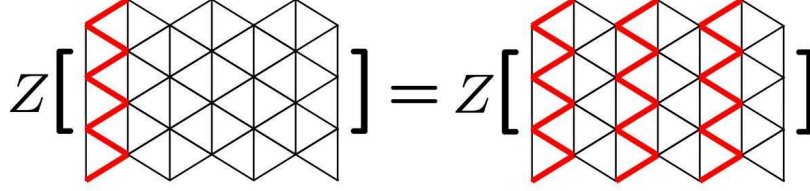


Figure 14: The partition function does not differ for any odd number of twists, owing to the Z_2 conservation of the twist.

Exact expressions for partition functions of a planar triangular Ising model with various boundary conditions were derived by Wu and Hu via ‘Grassmann path integral method’ [39]. Let Z_Λ ($Z_\Lambda^{(-)}$) denote the partition function with periodic boundary condition in both directions (with periodic in vertical and anti-periodic in horizontal direction), respectively. On a triangular lattice, three different couplings can be defined in each directions, so let us introduce J_1 as the coupling constant on vertical bonds in figs.10,11 and J_2, J_3 the other two. Reflection positivity however requires $J_2 = J_3 (\equiv J)$. Introduce

$$t_1 \equiv \tanh(J_1/k_B T), \quad t \equiv \tanh(J/k_B T). \quad (170)$$

Our convention is such that $t_{(1)} > 0$ corresponds to ferromagnetic coupling. On the (t, t_1) -plane, there is a line which corresponds to $T = T_c$ and we will call it the “critical line” in the following. Under the change of notation $L_1 \rightarrow N$ and $L_2 \rightarrow M$, the result due to Hu and Wu for this case reads

$$Z_\Lambda = \frac{1}{2} [2 \cosh^3(\beta J)]^{MN} \left[\Omega_{\frac{1}{2}, \frac{1}{2}} + \Omega_{\frac{1}{2}, 0} + \Omega_{0, \frac{1}{2}} - \text{sgn}(T - T_c) \Omega_{0, 0} \right], \quad (171)$$

$$Z_\Lambda^{(-)} = \frac{1}{2} [2 \cosh^3(\beta J)]^{MN} \left[\Omega_{\frac{1}{2}, \frac{1}{2}} + \Omega_{\frac{1}{2}, 0} - \Omega_{0, \frac{1}{2}} + \text{sgn}(T - T_c) \Omega_{0, 0} \right], \quad (172)$$

where

$$\Omega_{\mu\nu} = (A_0)^{MN/2} \prod_{p=0}^{M-1} \prod_{q=0}^{N-1} \left[1 - B \cos \frac{2\pi(p+\mu)}{M} - A \cos \frac{2\pi(q+\nu)}{N} - A \cos \left(\frac{2\pi(p+\mu)}{M} - \frac{2\pi(q+\nu)}{N} \right) \right]^{1/2}, \quad (173)$$

$$A_0 = (1 + t^2 t_1)^2 + (t_1 + t^2)^2 + 2t^2(1 + t_1)^2, \quad A = \frac{2(1 - t_1^2)(1 - t^2)t}{A_0}, \quad B = \frac{2t_1(1 - t^2)^2}{A_0} \quad (174)$$

with T_c the phase-transition temperature. Using the formulae given above, we can show

Lemma 6. *In the disordered phase ($\Leftrightarrow T > T_c$) we have*

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \left(1 - \frac{Z_\Lambda^{(-)}}{Z_\Lambda} \right)^{1/N} = e^{-\rho}, \quad (175)$$

where

$$\rho \equiv \cosh^{-1} \left(\frac{g(B)}{|A|} \right) > 0, \quad (176)$$

$$g(x) \equiv \begin{cases} \sqrt{-2x(1+x)} & \left(-1 < x < -\frac{1}{3} \right) \\ \frac{1-x}{2} & \left(-\frac{1}{3} \leq x < 1 \right) \end{cases}. \quad (177)$$

The order of two limits in (175) must not be changed.

In the above, $\frac{g(B)}{|A|} \geq 1$ and $|B| < 1$ are implicitly assumed; these can be shown for every $(t, t_1) \in (-1, 1)^2$ by elementary methods. (Note that $\frac{g(B)}{|A|} = 1$ defines the critical line.) The proof of theorem 6 is elementary but technically cumbersome, which we relegate to the appendix.

To gain an intuitive understanding of the above result, let us see fig.15, in which the projection of $1 - e^{-\rho}$ onto the (t_1, t) -plane is drawn. The black region corresponds to the (anti-)ferromagnetically ordered phase, while the colored region to the disordered phase. Brighter color represents larger ρ , hence larger mass gap. Fig.15 clearly shows a symmetry under $t \leftrightarrow -t$; this is a manifestation of the

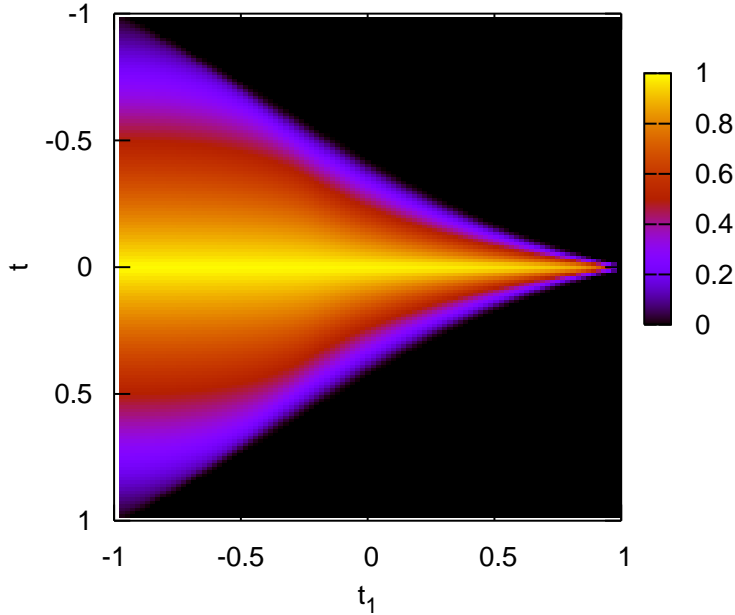


Figure 15: The projection of $1 - e^{-\rho}$ onto the (t_1, t) -plane. The black region corresponds to the (anti-)ferromagnetically ordered phase and the colored region to the disordered phase. Brighter (darker) color represents larger (smaller) ρ and especially ρ diverges on the $t = 0$ line. The boundary of the colored region signifies the critical line.

well-known fact that the triangular Ising model is invariant under simultaneous sign reversal of any two of J_1, J_2, J_3 .

The inequality of our primary interest, namely (101) for the triangular Ising model, reads²¹

$$\langle \sigma_0 \sigma_n \rangle_\Lambda \leq 2 \left\{ \frac{1}{2} \left(1 - \frac{Z_\Lambda^{(-)}}{Z_\Lambda} \right) \right\}^{n/N}. \quad (178)$$

Letting $M \rightarrow \infty$ after $N \rightarrow \infty$, we obtain

Theorem 6.

$$\langle \sigma_0 \sigma_n \rangle_\infty \leq 2e^{-\rho n}. \quad (179)$$

The above is the main result in this subsection; ρ is a rigorous lower bound of the true mass gap. We should keep in mind that the l.h.s. of (178) is a pair correlation between two spins on the same horizontal level as depicted in fig.10; the two spins are *not* on the same lattice axis.

Since the exponential decay rates are equal for both sides of the inequality in the square Ising model, it is natural to expect so in the triangular Ising model too. The asymptotic correlation between two spins *on the same lattice axis* in the triangular Ising model was derived by Stephenson for both ferromagnetic and antiferromagnetic couplings [40]. However, the asymptotic correlation between two spins *off the axis* is not found in the literature. We conjecture as follows:

Conjecture.

ρ is equal to the true off-axis mass gap for every (t, t_1) in the disordered phase.

The most straightforward way to test the conjecture would be to measure the mass gap directly via Monte Carlo simulation. However, as already seen from (169) the exponential falloff generically receives power law corrections (the so-called ‘Ornstein-Zernike’ decay [41]) which makes a reliable fitting difficult. To evade this hamper would call for sophisticated methods such as the Monte Carlo Transfer Matrix Method [42]. A numerical check of the conjecture therefore seems to be a highly nontrivial task, and we defer it to future work.

In a special case, analytical test is possible: when $t_1 = 0$ the model reduces to the isotropic square Ising model and the off-axis correlation function reduces to the *diagonal* correlation function. From (176) it follows that

$$\rho|_{t_1=0} = \cosh^{-1} \frac{(1+t^2)^2}{4|t|(1-t^2)} \quad \text{for } |t| < t_C = \sqrt{2} - 1, \quad (180)$$

which completely agrees with the exact *diagonal* mass gap obtained by Cheng and Wu in 1967 [36].

Further insight is gained by considering the isotropic case $t = t_1$. Since the two-point correlation function in this case is expected to be approximately isotropic (except for sign in antiferromagnetic case), it seems reasonable to compare $e^{-\rho}$ with $e^{-(\sqrt{3}/2)m}$ where m is the exact *on-axis* mass gap [40]. ($\sqrt{3}/2$ is a geometric correction factor.)

Fig.16 depicts the graphs of $e^{-\rho}$ and $e^{-(\sqrt{3}/2)m}$ against $t \in [-1, 2 - \sqrt{3}]$. For $t > 0$ they agree quite well; their nonzero difference is hardly discernible to the eye. For $t < 0$ agreement is still not bad. To say the least, the comparison suggests that ρ be fairly close to the true off-axis mass gap and supports, rather than defies, the conjecture.

It is readily seen from (177),(176) that $e^{-\rho}$ is a nonanalytic function in the region of negative t_1 , which, assuming the validity of our conjecture, implies non-analyticity of the mass gap. Such an exotic possibility definitely deserves further study. Strictly speaking, however, there are different possibilities that cannot be denied here: for example it could be the case that ρ equals the true mass

²¹ Actually, we originally proved (101) on a *square* lattice, but the whole procedure of the proof goes over to the case of a *triangular* lattice almost unchanged.

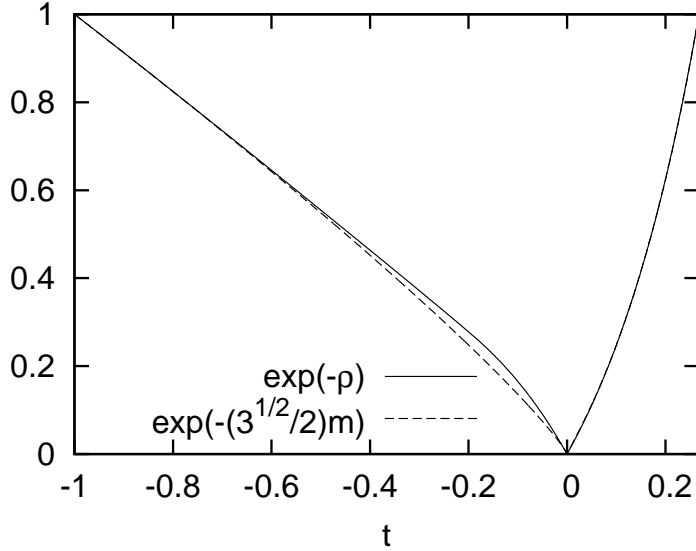


Figure 16: $e^{-\rho}$ and $e^{-(\sqrt{3}/2)m}$ are plotted against t in the case of isotropic case ($t = t_1$). Their agreement is remarkable especially at t positive.

gap *only* when $t_1 \geq 0$. In the latter case, non-analyticity of ρ does not signify that of the true mass gap.

Let us end this subsection by invoking the effectiveness of our approach. Although the circumstance concerning our conjecture is rather moot, it can be safely said that our result in this subsection is essentially new to the extent that it rigorously gives a lower bound for the still-unknown off-axis mass gap of the triangular (both ferromagnetic and antiferromagnetic, both isotropic and anisotropic) Ising model in the disordered phase.

3.6 Strong coupling analysis

In this section we show, using the convergent strong-coupling (taken as synonymous with high temperature) expansion, that both sides of the proved inequality (101) have an identical exponential decay rate at long distance as long as the on-axis correlation function is considered.²² The proof is valid in any dimension and makes no use of reflection positivity. Since the corresponding result in LGT has already been derived by Münster [28] (as mentioned in section 2.2) and since no essential difficulty arises in extending his proof to the case of spin models, we shall be brief here and only try to sketch the main idea behind the approach. Implications of this result to our conjecture will be discussed later.

For simplicity of exposition let us consider the isotropic square Ising model and its on-axis correlation function (though our argument is readily extendable to more general models such as PCM). A precise statement of the claim goes follows: as long as the size of the lattice is larger enough than n , the strong coupling expansion (SCE) of ρ and m are identical at least up to order n . (Our notation is such that the definition of ρ is in (175), m is the mass gap, $Z, Z^{(-)}, L_1, L_2$ are the same as in section 3.4 and $t \equiv \tanh(J/k_B T)$.)

²² High-temperature behavior of correlation functions in Ising-like models have been studied by many authors in a variety of methods; see ref.[43], for example. It is worthwhile to note that a majority of existing studies deal with neither the off-axis correlation function nor the case of an antiferromagnetic coupling. Hopefully a partial understanding of this fact will be gained through the discussions in this subsection.

Let us begin with the expression $Z = \sum_{\{\sigma\}} \prod_{i \neq j} (1 + t\sigma_i\sigma_j)$. Expanding Z into sums of disconnected loops and then taking the logarithm, we have $\log Z = \sum_{\gamma} t^{|\gamma|}$ with γ any connected loop and $|\gamma|$ the perimeter of γ . Using similar expression for $Z^{(-)}$ we obtain $\log \frac{Z^{(-)}}{Z} = -2 \sum_{\gamma \in S} t^{|\gamma|}$ where S is the set of loops wrapping around the lattice in x^1 -direction for odd number of times. Since we are interested in the limit $|t| \ll 1$, it is sufficient to consider only such loops that wind around the lattice in x^1 -direction only once. Factorizing the degeneracy factor due to translational symmetry in x^2 -direction, we have $\log \frac{Z^{(-)}}{Z} = -2L_2 \sum_{\gamma \in S'} t^{|\gamma|}$; the definition of S' should be obvious.

It is clear that the leading contribution, of order $O(t^{L_1})$, comes from a straight line extending in x^1 -direction while the subleading contributions come from loops which are formed via addition of some ‘decorations’ to the leading line. Dividing by the leading contribution and taking the logarithm will single out contributions of connected decorations, which is proportional to L_1 owing to the translational invariance of the straight line. Thus we find exactly the behavior (148) in section 3.3:

$$\log \left[\left(\frac{1}{L_2} \log \frac{Z^{(-)}}{Z} \right) / t^{L_1} \right] \propto L_1. \quad (181)$$

On the other hand, the on-axis two-point correlation function $\langle \sigma_x \sigma_{x+r} \rangle_{\infty}$ can be written as a sum over contributions of lines connecting σ_1 to σ_2 , whose leading term comes from a straight line extending between σ_1 and σ_2 and subleading terms from zig-zag lines that descend from the leading one through addition of decorations. In this way we see that the SCE of

$$\lim_{L_1, L_2 \rightarrow \infty} \log \left[\left(\frac{1}{L_2} \log \frac{Z^{(-)}}{Z} \right) / t^{L_1} \right] / L_1 \quad (= -\rho - \log t) \quad (182)$$

is identical, term by term, to the SCE of $\lim_{r \rightarrow \infty} \log [\langle \sigma_x \sigma_{x+r} \rangle_{\infty} / t^r] / r \quad (= -m - \log t)$. Hence $m = \rho$.

The argument above is valid for various other models as long as *on-axis* correlation functions are concerned. Then it is natural to ask about *off-axis* correlation functions. (This is the case relevant for the conjecture.) From fig.10 it is easily understood that the leading contribution to the SCE of $\frac{1}{L_2} \log \frac{Z^{(-)}}{Z}$ does not come from a single straight line: instead it comes from $\binom{L_1}{L_1/2}$ different loops, all of the same length L_1 . *So we now have a number of different ways to see a given higher-order loop as a sum of any one of the leading-order loops and a decoration added to it!* This implies that the counting of diagrams appearing in SCE of $\frac{1}{L_2} \log \frac{Z^{(-)}}{Z}$ (and of off-axis correlation function, too) is immensely complicated. We even face another problem: since most of the leading-order loops have no translational symmetry, it becomes a nontrivial task to show (181). For these reasons we cannot give a mathematically rigorous proof of the conjecture even at sufficiently high temperature.

Some caveats are in order.

- First, remember that we **did** confirm $m = \rho$ for the diagonal correlation function in the square Ising model ((180) and the accompanying discussion). This fact implies that our inability to prove (the very existence of m , ρ and) the equality $m = \rho$ for off-axis correlation function in SCE approach does not itself constitute a disproof.
- One may be tempted to argue that, since an exact one-to-one correspondence between the diagrams for SCE of $\langle \sigma_1 \sigma_2 \rangle_{\infty}$ and those for SCE of $\frac{1}{L_2} \log \frac{Z^{(-)}}{Z}$ exists, $m = \rho$ would readily

Furthermore we demonstrated our result explicitly in some solvable models and found in the square Ising model that the obtained lower bound of the mass gap is equal to the exact one. We also calculated the off-axis mass gap in the triangular Ising model for various couplings, but this time the exact mass gap is not known and direct comparison is impossible. We conjectured that the bound is indeed saturated and pointed out that the conjecture implies the non-analyticity of the mass gap. We have tested its validity in several ways, including strong coupling analysis, but a definitive conclusion is still lacking and is left for future work.

At present the mechanisms of the quark confinement in non-Abelian gauge theories and the mass gap generation in non-Abelian spin models still remain elusive, and we hope that our result will be useful for further clarification of the issue.

Note added

After this work was completed, we learned that C. Borgs and E. Seiler had already obtained a result very similar to theorem 2 of this paper; see Lemma II.8 and the accompanying discussion in ref.[44]²⁴. But since it links the Polyakov loop correlator and not the Wilson loop with the electric flux free energy, it is not quite the same as ours. It does, however, already imply the 't Hooft's string tension is less than or equal to Wilson's (see (II.48) and (II. 50) of ref.[44]). Finally we note that their results hardly overlap with ours in section 3.

Acknowledgment

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Appendix A Proof of Lemma 6

For $T > T_c$, (171) and (172) yield

$$\frac{Z_\Lambda - Z_\Lambda^{(-)}}{Z_\Lambda + Z_\Lambda^{(-)}} = \frac{\Omega_{0,\frac{1}{2}} - \Omega_{0,0}}{\Omega_{\frac{1}{2},\frac{1}{2}} + \Omega_{\frac{1}{2},0}}. \quad (183)$$

Let us define $\theta_A^B(\mu, p, M) \geq 0$ by

$$\cosh \theta_A^B(\mu, p, M) \equiv \left| 1 - B \cos \frac{2\pi(p+\mu)}{M} \right| \left/ \left| 2A \cos \frac{\pi(p+\mu)}{M} \right| \right|. \quad (184)$$

It is tedious but straightforward to show that the minimum of $\cosh \theta_A^B$ as a function of $-1 \leq \cos \frac{\pi(p+\mu)}{M} \leq 1$ is given by $\frac{g(B)}{|A|}$ (see (177)), and that $\frac{g(B)}{|A|} \geq 1$ for every $(t, t_1) \in (0, 1)^2$, with equality on the critical line. After elementary calculations, we find

$$(\Omega_{\mu\nu})^2 \simeq \left(\frac{A_0}{2} \right)^{MN} \left\{ \prod_{p=0}^{M-1} \left| 2A \cos \frac{\pi(p+\mu)}{M} \right| \exp \theta_A^B(\mu, p, M) \right\}^N \quad (185)$$

²⁴ I thank E. Seiler for kindly pointing out this fact to me.

for $N \gg 1$. Since (185) has no dependence on ν ,

$$\lim_{N \rightarrow \infty} \frac{\Omega_{\frac{1}{2},0}}{\Omega_{\frac{1}{2},\frac{1}{2}}} = \lim_{N \rightarrow \infty} \frac{\Omega_{0,0}}{\Omega_{0,\frac{1}{2}}} = 1. \quad (186)$$

Next, using (185) we get

$$\frac{\Omega_{\frac{1}{2},\frac{1}{2}}}{\Omega_{0,\frac{1}{2}}} \simeq \exp \left\{ N \sum_{k=0}^{2M-1} (-1)^{k+1} f_A^B \left(\frac{k}{2M} \right) \right\} \quad \text{for } N \gg 1, \quad (187)$$

with

$$f_A^B(x) \equiv \frac{1}{2} \log \left\{ 1 - B \cos(2\pi x) + \sqrt{(1 - B \cos(2\pi x))^2 - (2A \cos(\pi x))^2} \right\}. \quad (188)$$

Let us investigate how fast $\frac{\Omega_{0,0}}{\Omega_{0,\frac{1}{2}}}$ converges to 1. Using (185) we can show

$$\left(\frac{\Omega_{0,0}}{\Omega_{0,\frac{1}{2}}} \right)^2 \simeq \prod_{p=0}^{M-1} \left\{ 1 - 4 \exp(-N \theta_A^B) \right\} \quad \text{for } N \gg 1. \quad (189)$$

Define $\bar{\theta}$ as the smallest of $\{\theta_A^B(0, p, M)\}_p$. Then (189) simplifies to

$$\left(\frac{\Omega_{0,0}}{\Omega_{0,\frac{1}{2}}} \right)^2 \simeq 1 - K e^{-N \bar{\theta}} \quad \text{for } N \gg 1. \quad (190)$$

K is an integer $\in \{4, 8, 12, 16\}$, dependent on A , B and M . Substitution of (186), (187) and (190) into (183) yields

$$1 - \frac{Z_{\Lambda}^{(-)}}{Z_{\Lambda}} \simeq 2 \exp \left\{ -N \left[\bar{\theta} + \sum_{k=0}^{2M-1} (-1)^{k+1} f_A^B \left(\frac{k}{2M} \right) \right] \right\} \quad \text{for } N \gg 1. \quad (191)$$

Since $\bar{\theta} = \cosh^{-1} \left(\frac{g(B)}{|A|} \right) + O\left(\frac{1}{M}\right)$ and $\sum_{k=0}^{2M-1} (-1)^{k+1} f_A^B \left(\frac{k}{2M} \right) = O\left(\frac{1}{M}\right)$ for $M \gg 1$, we find

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \left(1 - \frac{Z_{\Lambda}^{(-)}}{Z_{\Lambda}} \right)^{1/N} = e^{-\rho}, \quad (192)$$

$$\rho \equiv \cosh^{-1} \left(\frac{g(B)}{|A|} \right), \quad (193)$$

which is the desired result. \square

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